Math 405: Lie Algebras
Exercise Set 7

Let $L = \mathfrak{sl}(2, \mathbb{C})$ throughout.

In class, we defined an L-module structure on $V = \mathfrak{sl}(3, \mathbb{C})$ as follows. For $z \in L$, let

$$\hat{z} = \left[\begin{array}{c|c} z & 0 \\ \hline 0 & 0 \end{array}\right].$$

Then for all $v \in V$, we define $z \cdot v = [\hat{z}, v]$.

We can then decompose V into a direct sum of irreducible submodules:

$$\begin{aligned} V &= \mathrm{sp}\{\hat{e}, \hat{h}, \hat{f}\} \oplus \mathrm{sp}\{e_{13}, e_{23}\} \oplus \mathrm{sp}\{e_{31}, e_{32}\} \oplus \mathrm{sp}\{\hat{I}_2 - 2e_{33}\} \\ &\cong V_2 \oplus V_1 \oplus V_1 \oplus V_0. \end{aligned}$$

(You should verify that L acts trivially $\mathrm{sp}\{\hat{I}_2 - 2e_{33}\}$, but you do not need to write this part up.)

1. Now let $V = V_3 \otimes V_7$. We once again wish to decompose V into a direct sum of irreducible submodules.
   (a) Verify that elements of the form $x^a y^b \otimes x^j y^k$ are eigenvectors of h, and determine the associated eigenvalues.
   (b) Determine $\dim(W_0) + \dim(W_1)$, i.e. find the number of independent eigenvectors with eigenvalues 0 or 1. Note that this is the number of distinct irreducible submodules.
   (c) Find the highest weight vectors for each irreducible submodule.
   (d) Write V as a direct sum of the form $V_{d_1} \oplus \cdots \oplus V_{d_k}$.
   (e) (Optional Bonus Problem:) Develop a general formula for the decomposition of $V_m \otimes V_n$.