Math 405: Lie Algebras
Exercise Set 6

1. Show that V is irreducible if and only if for any non-zero $v \in V$ the submodule generated by v contains all elements of V. The submodule generated by v is defined to be the subspace of V spanned by all elements of the form $x_1 \cdot (x_2 \cdot \ldots (x_m \cdot v) \ldots)$, where $x_1, \ldots, x_m \in L$.

2. Let L be a finite-dimensional Lie algebra. Let V be L with the L-module structure on V given by the adjoint representation of L. Show that the submodules of V are precisely the ideals of L.

3. Let L be a complex Lie algebra and let V be a finite dimensional irreducible L-module.

   (a) If $z \in Z(L)$, then z acts by scalar multiplication on V; that is, there is some $\lambda \in \mathbb{C}$ such that $z \cdot v = \lambda v$ for all $v \in V$. (*Hint:* Schur's Lemma)

   (b) If L is solvable, show that $\dim(V)$ must be 1. (*Hint:* Lie's Theorem)

4. Let L be the 2-dimensional complex Lie algebra with basis x, y such that $[x, y] = x$. Check that we may define a representation of L on $\mathbb{C}^2$ by setting

$$\phi(x) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \phi(y) = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}.$$

   Show that this representation is isomorphic to the adjoint representation of L on itself.