Math 405: Lie Algebras
Exercise Set 5

1. Show that if L is nilpotent, then the Killing form of L is identically zero.

2. Let L be the solvable 2-dimensional Lie algebra with basis $\{x, y\}$ defined by the bracket $[x, y] = x$. Show that L has nontrivial Killing form.

3. Prove that L is solvable if and only if $[L, L]$ lies in the radical of the Killing form, ie the set $S = \{x \in L \mid \kappa(x, y) = 0 \ \forall y \in L\}$.

4. Let $(x_1, \ldots, x_n)$ be an ordered basis of a Lie algebra L. Then the Killing form of L can be described by an $n \times n$ symmetric matrix $B = [\kappa(x_i, x_j)]$.

   (a) Show that κ is nondegenerate if and only if $\det(B) \neq 0$.

   (b) Let $L = \mathfrak{sl}(2, \mathbb{F})$ with the usual basis (x, h, y). Find the matrix of the Killing form of L, and show that the Killing form is nondegenerate.

   (c) Let $L = \mathfrak{gl}(2, \mathbb{F})$ with the usual basis $(e_{11}, e_{12}, e_{21}, e_{22})$. Find the matrix of the Killing form of L, and show that the Killing form is degenerate.