Math 405: Lie Algebras
Exercise Set 4

1. In this exercise, we prove that every nilpotent Lie algebra L has an outer derivation δ, as follows. Suppose L is nilpotent, and write $L = K + \mathbb{F}z$, where K is an ideal of codimension 1.

   (a) Show that the *centralizer* of K in L, given by
   $$C_L(K) = \{x \in L \mid [x, y] = 0 \quad \forall y \in K\}$$
   is nonzero.

   (b) Choose n such that $C_L(K) \subseteq L^n$ and $C_L(K) \not\subset L^{n+1}$. Let $z_0 \in C_L(K) - L^{n+1}$. Show that the linear map δ sending K to 0 and z to z_0 is a derivation of L.

   (c) Show that δ cannot be written as $\mathrm{ad}(x)$ for any $x \in L$.

2. Let L be a complex Lie algebra. Show that L is nilpotent if and only if every 2-dimensional subalgebra of L is abelian. (*Note*: You may use the fact that if a matrix over $\mathbb{C}$ has only 0 as its eigenvalue, then it is nilpotent. This is also a hint for how to approach the reverse direction.)

3. Let L be a complex Lie algebra. Show that L is solvable if and only if L' is nilpotent.

4. Let $x, y : V \to V$ be linear maps from a complex vector space V to itself. Suppose that x and y both commute with $[x, y]$. Then $[x, y]$ is a nilpotent map.

5. This exercise illustrates the failure of Lie's Theorem when $\mathbb{F}$ is allowed to have prime characteristic p. Suppose p is prime and let $\mathbb{F} = \mathbb{Z}_p$ (the field of integers mod p). Consider the following $p \times p$ matrices:
   $$x = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad y = \mathrm{diag}(0, 1, 2, 3, \ldots, p-1).$$

   (a) Check that $[x, y] = x$, and deduce that x and y span a 2-dimensional solvable subalgebra L of $\mathfrak{gl}(p, \mathbb{Z}_p)$.

   (b) Show that x and y have no common eigenvector.

   (c) Show that $x \in L'$ is *not* nilpotent in this case.