Math 405: Lie Algebras
Exercise Set 3

1. Show that L is solvable if and only if there exists a chain of subalgebras

$$L = I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots \supseteq I_k = 0$$

such that I_{j+1} is an ideal of I_j and such that each quotient I_j/I_{j+1} is abelian. (*Hint*: Suppose that I is an ideal of L. Show that L/I is abelian if and only if I contains the derived algebra L'.)

2. Let L be a Lie algebra. Recall that $\mathrm{ad}(L)$ is the subalgebra of $\mathfrak{gl}(L)$ consisting of inner derivations.

   (a) Show that L is solvable if and only if $\mathrm{ad}(L)$ is solvable.

   (b) Show that L is nilpotent if and only if $\mathrm{ad}(L)$ is nilpotent.

3. Show that a Lie algebra is semisimple if and only if it has no nonzero abelian ideals. (This was the original definition of semisimplicity given by Wilhelm Killing.)

4. Prove that the sum of two nilpotent ideals of a Lie algebra L is again a nilpotent ideal. (Note that this implies that L posesses a unique maximal nilpotent ideal.) *Hint:* Show by induction that

$$(I + J)^n \subseteq (I^n + I^{n-1} \cap J^0 + I^{n-2} \cap J^1 + \cdots + I^0 \cap J^{n-1} + J^n).$$

5. Show by counterexample that it is possible to have a Lie algebra L with ideal I such that L/I and I are nilpotent but L is not.

6. Let L be a 3-dimensional Lie algebra with basis $\{x, y, z\}$ defined by the brackets:

$$[x, y] = z, [x, z] = y, [y, z] = 0.$$

   (a) Verify that this satisfies the Jacobi identity applied to the three basis elements.

   (b) Show that L is solvable.

   (c) Show that L is not nilpotent.

   (d) Determine the unique maximal nilpotent ideal.