Math 405: Lie Algebras
Exercise Set 2

1. Let L be a Lie algebra and $x \in L$. Prove that the subspace of L spanned by the eignevectors of $\mathrm{ad}(x)$ is a subalgebra.

2. Show that the set of all inner derivations $\mathrm{ad}(x)$, $x \in L$, is an ideal of $\mathrm{Der}(L)$.

3. Let L_1 and L_2 be Lie algebras, and let $\phi : L_1 \to L_2$ be a surjective Lie algebra homomorphism. True or false:

   (a) $\phi([L_1, L_1]) = [L_2, L_2]$

   (b) $\phi(Z(L_1)) = Z(L_2)$

   (c) If $h \in L_1$ and $\mathrm{ad}(h)$ is diagonalisable then $\mathrm{ad}(\phi(h))$ is diagonalisable.

   What if ϕ were an isomorphism instead?

4. Suppose L_1 and L_2 are Lie algebras. Let $L = \{(x_1, x_2) : x_i \in L_i\}$ be the direct sum of their underlying vector spaces.

   (a) Show that if we define
   $$[(x_1, x_2), (y_1, y_2)] = ([x_1, y_1], [x_2, y_2]),$$
   then L becomes a Lie algebra, the direct sum of L_1 and L_2, denoted $L = L_1 \oplus L_2$.

   (b) Show that if $L = L_1 \oplus L_2$, then L has ideals isomorphic to L_1 and L_2.

   (c) Prove that $\mathfrak{gl}(2, \mathbb{C})$ is isomorphic to the direct sum of $\mathfrak{sl}(2, \mathbb{C})$ with $\mathbb{C}$.

5. Let L be the 3-dimensional complex Lie algebra with basis $\{x, y, z\}$. and Lie bracket defined by $[x, y] = z, [y, z] = x, [z, x] = y$. (Here L is the "complexification" of the Lie algebra $\mathbb{R}^3$ under the cross product.)

   (a) Show that L is isomorphic to the Lie subalgebra of $\mathfrak{gl}(3, \mathbb{C})$ consisting of all 3×3 antisymmetric matrices with entries in $\mathbb{C}$.

   (b) Find an explicit isomorphism showing $\mathfrak{sl}(2, \mathbb{C}) \equiv L$.

6. Show that over $\mathbb{R}$ the Lie algebras $\mathfrak{sl}(2, \mathbb{R})$ and $\mathbb{R}^3$ under the cross-product are not isomorphic. Hint: Prove that there is no non-zero $x \in \mathbb{R}^3$ such that the map $\mathrm{ad}(x)$ is diagonalisable.