Math 405: Lie Algebras
Exercise Set 1

1. Consider $\mathbb{R}^3$ as a Lie algebra with the bracket given by the cross-product, i.e.

$$[u, v] = u \times v.$$

   Show that this satisfies the Jacobi identity.

2. Consider the general linear algebra $\mathfrak{gl}(V)$ under the bracket given by

$$[x, y] = x \circ y - y \circ x.$$

   Show that this satisfies the Jacobi identity.

3. Show that the following subsets of $\mathfrak{gl}(n, \mathbb{F})$ are Lie algebras (with the same bracket operation as in $\mathfrak{gl}(n, \mathbb{F})$).

   (a) $\mathfrak{b}(n, \mathbb{F})$, the set of upper triangular matrices

   (b) $\mathfrak{d}(n, \mathbb{F})$, the set of diagonal matrices

   (c) $\{x \in \mathfrak{gl}(n, \mathbb{F}) \mid sx = -x^t s\}$, where s is a fixed $n \times n$ matrix and x^t denotes the transpose of x.

4. A *derivation* of an algebra A is a linear function $D : A \to A$ such that for all $x, y \in A$, we have $D(xy) = D(x)y + xD(y)$.

   (a) Show that the composition of two derivations is not necessarily a derivation.

   (b) Show that the commutator of two derivations is a derivation.

5. Compute the exponential of the following matrices:

   (a) $X = \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix}$

   (b) $Y = \begin{bmatrix} 0 & 3 & 5 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

   (c) $Z = \begin{bmatrix} 3 & -1 \\ 0 & 3 \end{bmatrix}$