Restriction:

Defn: Let H be a subgp of G. Then any $\mathbb{C}[G]$-module V can be regarded as a $\mathbb{C}[H]$-module, in which case we denote it by $\mathrm{Res}_H V$ or $V\downarrow H$, and call it the restriction of V to H. If V has char χ then $V\downarrow H$ has char $\chi\downarrow H$.

Note: If χ is irred, it does not necessarily follow that $\chi\downarrow H$ is irred.

Prop: Let H be a subgp of G and let ψ be a nonzero char. of H. Then there exists an irred char χ of G such that $\langle \chi\downarrow H, \psi\rangle_H \neq 0$.

Pf: Let $\chi_1, \dots, \chi_k$ be the irred. chars of G. Let φ be the regular char of G. Then we know $\varphi = \sum_{i=1}^{k} \chi_i(e)\chi_i$.

Now observe that the regular module $\mathbb{C}[G]$ of G contains $\mathbb{C}[H]$ as a subspace, and $\mathbb{C}[H]$ is closed under the action of H. (Thus, $\mathrm{Res}_H\, \mathbb{C}[H] = \mathbb{C}[H]$ is a $\mathbb{C}[H]$-submodule of $\mathrm{Res}_H\, \mathbb{C}[G]$.)

Assume ψ is irreducible. Then the corresponding $\mathbb{C}[H]$-module V is a submodule of $\mathbb{C}[H]$. Therefore $\langle \varphi, \psi\rangle \neq 0$ so

$$0 \neq \left\langle \sum_{i=1}^{k} \chi_i(e)\chi_i, \psi \right\rangle = \sum_{i=1}^{k} \chi_i(e)\langle \chi_i, \psi\rangle$$

so $\langle \chi_i, \psi\rangle \neq 0$ for some i.

We can extend this to reducible ψ by noting that for some constituent ψ_j of ψ, $\exists\, i$ such that $\langle \chi_i, \psi_j\rangle \neq 0$. $\square$

Thm: (Clifford's Thm) Suppose $H \trianglelefteq G$, χ is an irred char of G, and $\psi_1, \dots, \psi_m$ are the constituents of $\chi \downarrow H$. Then:

1) the ψ_i all have the same degree

2) $\langle \chi \downarrow H, \psi_i \rangle$ is the same for all ψ_i

Pf: 1) Let V be a $\mathbb{C}[G]$-mod with char χ. Let U be an irred $\mathbb{C}[H]$-submod of $V \downarrow H$. Then $\forall g \in G$, the set $g \cdot U = \{g \cdot \vec{u} \mid \vec{u} \in U\}$ is a subspace of V and $\forall h \in H$:

$h(g \cdot \vec{u}) = (hg) \cdot \vec{u} = g(g^{-1}hg) \cdot \vec{u} \in g \cdot U$

since $g^{-1}hg \in H$, so $(g^{-1}hg) \cdot \vec{u} \in U$. Thus, $g \cdot U$ is also a $\mathbb{C}[H]$-submod of $V \downarrow H$.

Now suppose W is a submod of gU. Then $g^{-1}W$ is a submod of U. Since U is irred, $g^{-1}W$ is U or $\{0\}$, and so W is either $g \cdot U$ or $\{0\}$, and hence $g \cdot U$ is also irred. Also, since g is an invertible linear trans, $\dim g \cdot U = \dim U$.

Finally, note that $\sum_{g \in G} gU$ is a $\mathbb{C}[G]$-submod of V. Since V is irred, $\sum_{g \in G} gU = V$, so V is the direct sum of some of the irred $\mathbb{C}[H]$-modules $g \cdot U$. Thus $V = g_1 U \oplus \cdots \oplus g_k U$, and so the constituents of χ all have the same degree.

2) Let $\langle \chi \downarrow H, \psi_1 \rangle = d$. Then V has a $\mathbb{C}[H]$-submod X_1 such that X_1 is the direct sum of d copies of an irred submod U_1. But we know $U_i = gU_1$ for some g, so gX_1 must decompose into d copies of U_i, and thus $\langle \chi \downarrow H, \psi_i \rangle = d$ for all i. $\square$

Prop: Let $H \leq G$, let χ be an irred char of G, and let $\psi_1, \ldots, \psi_r$ be the irred chars of H. Then $\chi \downarrow H = d_1 \psi_1 + \cdots + d_r \psi_r$, where

$$\sum_{i=1}^{r} d_i^2 \leq [G:H]$$

Equality holds iff $\chi(g) = 0 \ \forall g \in G - H$

Pf: We observe that:

$$\sum_{i=1}^{r} d_i^2 = \langle \chi\downarrow H, \chi\downarrow H \rangle_H = \frac{1}{|H|} \sum_{h \in H} \chi(h)\chi(h)^*$$

Since χ is irred, we have:

$$1 = \langle \chi, \chi \rangle_G = \frac{1}{|G|}\left(\sum_{h \in H} \chi(h)\chi(h)^* + \sum_{g \in G-H} \chi(g)\chi(g)^* \right)$$

$$= \frac{|H|}{|G|}\left(\sum_{i=1}^{r} d_i^2 \right) + K$$

where $K > 0$. The result follows. □

Normal Subgroups of Index 2:

Prop: Suppose $[G:H] = 2$. Then $H \trianglelefteq G$.

Prop: Let H be a subgp of index 2 in G, and let χ be an irred char of G. Then either

1.) $\chi \downarrow H$ is irred, or

2.) $\chi \downarrow H$ is the sum of two distinct irred chars of H of the same degree.

Pf: Let $\chi = d_1\psi_1 + \cdots + d_k\psi_k$, where $\psi_1, \ldots, \psi_k$ are the irred chars of H. Then $\sum d_i^2 \leq 2$, so either $\chi = \psi_i$ or $\chi = \psi_i + \psi_j$ for some i, j with $i \neq j$. In the latter case, ψ_i and ψ_j have the same degree by Clifford's Theorem. □

Prop: Let $[G:H]=2$, and let χ be an irred char. of G. Then TFAE:

1) $\chi \downarrow H$ is irred
2) $\chi(g) \neq 0$ for some $g \in G-H$.
3) $\chi \neq \chi\lambda$, where λ is the lift of the non-trivial char of G/H.

Note: $\lambda(g) = \begin{cases} 1 & \text{if } g \in H \\ -1 & \text{if } g \notin H \end{cases}$

Prop: Let $[G:H]=2$, and let χ, φ be irred chars of G.

1) If $\chi \downarrow H$ is irred, and $\varphi \downarrow H = \chi \downarrow H$, then $\varphi = \chi$ or $\varphi = \lambda\chi$.
2) If $\chi \downarrow H = \psi_1 + \psi_2$ where the ψ_i are irred, and $\varphi \downarrow H$ has ψ_1 or ψ_2 as a constituent, then $\varphi = \chi$.

Pf: 1) We have:

$$(\chi + \chi\lambda)(g) = \begin{cases} 2\chi(g) & \text{if } g \in H \\ 0 & \text{if } g \notin H, \end{cases}$$

so

$$\langle \chi + \chi\lambda, \varphi \rangle_G = \frac{1}{|G|}\left(\sum_{g \in H} 2\chi(g)\varphi(g)^*\right)$$

$$= \frac{1}{H} \sum_{g \in H} \chi(g)\varphi(g)^* = \langle \chi \downarrow H, \varphi \downarrow H \rangle_H = 1$$

Thus, either $\varphi = \chi$ or $\varphi = \lambda\chi$.

2) We have $\chi(g) = 0 \ \forall g \notin H$, so

$$\langle \chi, \varphi \rangle_G = \frac{1}{|G|} \sum_{g \in H} \chi(g)\varphi(g)^* = \frac{1}{2}\langle \chi \downarrow H, \varphi \downarrow H \rangle_H \neq 0$$

so $\varphi = \chi$. $\square$

Example:

$G = S_5$

g_i	e	$(1\ 2)$	$(1\ 2)(3\ 4)$	$(1\ 2\ 3)$	$(1\ 2\ 3)(4\ 5)$	$(1\ 2\ 3\ 4)$	$(1\ 2\ 3\ 4\ 5)$
$\lvert \bar{g}_i \rvert$	1	10	15	20	20	30	24
χ_1	1	1	1	1	1	1	1
χ_2	1	-1	1	1	-1	-1	1
χ_3	4	2	0	1	-1	0	-1
χ_4	4	-2	0	1	1	0	-1
χ_5	5	1	1	-1	1	-1	0
χ_6	5	-1	1	-1	-1	1	0
χ_7	6	0	-2	0	0	0	1

$H = A_5$

(Note that $\chi_i \downarrow H$ is irred iff $\chi_i(g) = 0 \ \forall g \notin H$.)

g_i	e	$(1\ 2)(3\ 4)$	$(1\ 2\ 3)$	$(1\ 2\ 3\ 4\ 5)$	$(2\ 1\ 3\ 4\ 5)$
$\lvert \bar{g}_i \rvert$	1	15	20	12	12
$\psi_1 = \chi_1 \downarrow G$	1	1	1	1	1
$\psi_2 = \chi_3 \downarrow G$	4	0	1	-1	-1
$\psi_3 = \chi_5 \downarrow G$	5	1	-1	0	0
χ_7: ψ_4	3	-1	0	$\frac{1}{2}(1+\sqrt{5})$	$\frac{1}{2}(1-\sqrt{5})$
ψ_5	3	-1	0	$\frac{1}{2}(1-\sqrt{5})$	$\frac{1}{2}(1+\sqrt{5})$

(The last two rows were completed using orthogonality relations.)

Induction:

Let $H \leq G$, and let $g_1, \ldots, g_k$ be representatives of the left cosets of H in G. Then given a $\mathbb{C}[H]$-module W, we wish to define a $\mathbb{C}[G]$-module V with the form $g_1 W \oplus \cdots \oplus g_k W$.

Defn: Let $H \leq G$. Let U be a $\mathbb{C}[H]$-submodule of $\mathbb{C}[H]$. Then $U\uparrow G = \{g \cdot u \mid g \in G, u \in U\}$, and we call $U \uparrow G$ (or $\mathrm{Ind}_H^G(U)$) the $\mathbb{C}[G]$-module induced from U.

Example: $G = D_3 = \langle a, b \mid a^3 = b^2 = e, \ ab = ba^{-1} \rangle$
$H = C_3 = \langle a \mid a^3 = e \rangle$
The irred. $\mathbb{C}[H]$-submods of $\mathbb{C}[H]$ are:
$W_1 = \mathrm{span}\{e + a + a^2\}$
$W_2 = \mathrm{span}\{e + \omega^2 a + \omega a^2\}$
$W_3 = \mathrm{span}\{e + \omega a + \omega^2 a^2\}$
These induce the following $\mathbb{C}[G]$-modules:
$W_1 \uparrow G = \mathrm{span}\{e + a + a^2, \ b + ba + ba^2\} = U_1 \oplus U_2$
$W_2 \uparrow G = \mathrm{span}\{e + \omega^2 a + \omega a, \ b + \omega^2 ba + \omega ba^2\} = U_3$
$W_3 \uparrow G = \mathrm{span}\{e + \omega a + \omega^2 a, \ b + \omega ba + \omega^2 ba^2\} = U_4$

Prop: Let $H \leq G$, and let U be a $\mathbb{C}[H]$-submod of $\mathbb{C}[H]$.
1) If V is a $\mathbb{C}[H]$-submod of $\mathbb{C}[H]$ that is $\mathbb{C}[H]$-isomorphic to U, then $V \uparrow G$ is $\mathbb{C}[G]$-isomorphic to $U \uparrow G$.
2) If $U = U_1 \oplus \cdots \oplus U_m$, then $U \uparrow G = (U_1 \uparrow G) \oplus \cdots \oplus (U_m \uparrow G)$.

Defn: Let U be a $\mathbb{C}[H]$-module. Then $U = U_1 \oplus \cdots \oplus U_r$, where the U_i are irreducible $\mathbb{C}[H]$-submodules of U, and each U_i is $\mathbb{C}[H]$-isomorphic to a submod. V_i of $\mathbb{C}[H]$. Then $U \uparrow G = (U_1 \uparrow G) \oplus \cdots \oplus (U_r \uparrow G)$, where $U_i \uparrow G \cong_{\mathbb{C}[G]} V_i \uparrow G$.

Defn: Let $H \leq G$, and let U be a $\mathbb{C}[H]$-module. Then if U has character ψ, we denote the character of $U\uparrow G$ by $\psi\uparrow G$.

Thm: (Frobenius Reciprocity Thm:) Let ψ be a character of H and let χ be a character of G. Then:
$\langle \psi\uparrow G, \chi \rangle_G = \langle \psi, \chi\downarrow G \rangle_H$.

Prop: Let ψ be a character of H. Then let $\dot{\psi}: G \to \mathbb{C}$ be the function given by:

$$\dot{\psi}(g) = \begin{cases} \psi(g) & \text{if } g \in H \\ 0 & \text{if } g \notin H \end{cases}$$

Then

$$\psi\uparrow G(g) = \frac{1}{|H|} \sum_{y \in G} \dot{\psi}(ygy^{-1}).$$