Tensor Products

Defn: Let V and W be vector spaces over $\mathbb{C}$ with bases $\vec{v}_1, \ldots, \vec{v}_n$ and $\vec{w}_1, \ldots, \vec{w}_m$, respectively. Then the tensor product $V \otimes W$ is the vector space over $\mathbb{C}$ with basis given by $\{\vec{v}_i \otimes \vec{w}_j \mid 1 \le i \le n, 1 \le j \le m\}$

Defn: For $\vec{v} \in V$, $\vec{w} \in W$, we define the tensor $\vec{v} \otimes \vec{w} \in V \otimes W$ by:
$$\vec{v} \otimes \vec{w} = \left(\sum_i \lambda_i \vec{v}_i\right) \otimes \left(\sum_j \mu_j \vec{w}_j\right) = \sum_{i,j} \lambda_i \mu_j (\vec{v}_i \otimes \vec{w}_j)$$

Prop: For $\vec{v} \in V$, $\vec{w} \in W$, $\lambda \in \mathbb{C}$, we have:
$$\vec{v} \otimes (\lambda \vec{w}) = (\lambda \vec{v}) \otimes \vec{w} = \lambda(\vec{v} \otimes \vec{w})$$

Defn: Let V, W be $\mathbb{C}[G]$-modules. Then we can define an action of G on $V \otimes W$ by:
$$g \cdot (\vec{v}_i \otimes \vec{w}_j) = (g \cdot \vec{v}_i) \otimes (g \cdot \vec{w}_j)$$

Note: For all $\vec{v} \in V$, $\vec{w} \in W$, $g \in G$, we have:
$$g \cdot (\vec{v} \otimes \vec{w}) = (g \cdot \vec{v}) \otimes (g \cdot \vec{w})$$
However, it is not true that for all $r \in \mathbb{C}[G]$:
$$r \cdot (\vec{v} \otimes \vec{w}) = (r \cdot \vec{v}) \otimes (r \cdot \vec{w}).$$

Prop: Let V and W be $\mathbb{C}[G]$-modules with characters χ and ψ, resp. Then the char of $V \otimes W$ (as a $\mathbb{C}[G]$-module) is $\chi\psi$, where $\chi\psi(g) = \chi(g)\psi(g)$.

Cor: The product of two characters of G is again a character of G.

Examples:

Let $G = D_3$ and let V and W be $\mathbb{C}[G]$-modules corresponding to the sign and tautological representations, respectively. Then $\dim V = 1$ and $\dim W = 2$. Let $\vec{v}$ be the basis vector of V and $\vec{w}_1, \vec{w}_2$ be the basis of W. For each of the following spaces, we give a basis and the matrices of $f, g \in D_3$ relative to that basis.

1. $V \oplus W = \text{span}\{\vec{v}, \vec{w}_1, \vec{w}_2\}$

$$f \longmapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix}, \quad g \longmapsto \left[\begin{array}{c|cc} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right]$$

2. $V \otimes W = \text{span}\{\vec{v} \otimes \vec{w}_1, \vec{v} \otimes \vec{w}_2\}$

$$f \longmapsto \begin{bmatrix} \omega & 0 \\ 0 & \omega^2 \end{bmatrix}, \quad g \longmapsto \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

3. $W \oplus W = \text{span}\{\vec{w}_1, \vec{w}_2, \vec{w}_1', \vec{w}_2'\}$

$$f \longmapsto \left[\begin{array}{cc|cc} \omega & 0 & 0 & 0 \\ 0 & \omega^2 & 0 & 0 \\ \hline 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & \omega^2 \end{array}\right], \quad g \longmapsto \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

4. $W \otimes W = \text{span}\{\vec{w}_1 \otimes \vec{w}_1, \vec{w}_1 \otimes \vec{w}_2, \vec{w}_2 \otimes \vec{w}_1, \vec{w}_2 \otimes \vec{w}_2\}$

$$f \longmapsto \begin{bmatrix} \omega^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega \end{bmatrix}, \quad g \longmapsto \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

In each case, we can verify that:

$\chi_{\rho \oplus \sigma} = \chi_\rho + \chi_\sigma$

$\chi_{\rho \otimes \sigma} = \chi_\rho \cdot \chi_\sigma$.

Decomposing $V \otimes V$:

Let V be a $\mathbb{C}[G]$-mod with char χ.
Then $V \otimes V$ is a $\mathbb{C}[G]$-mod with char χ^2.
Let $\vec{v}_1, \dots, \vec{v}_n$ be a basis for V. Then define

$$\varphi : V \otimes V \to V \otimes V$$
$$\vec{v}_i \otimes \vec{v}_j \mapsto \vec{v}_j \otimes \vec{v}_i$$

Thus $\varphi(\vec{v} \otimes \vec{w}) = \vec{w} \otimes \vec{v}$ $\forall \vec{v}, \vec{w} \in V$, so φ is independent of basis. We also observe $\varphi^2 = 1_{V \otimes V}$.

We define the following subspaces of $V \otimes V$:

$$Sym^2(V) = S(V \otimes V) = \{\vec{x} \in V \otimes V \mid \varphi(\vec{x}) = \vec{x}\}$$
$$Alt^2(V) = A(V \otimes V) = \{\vec{x} \in V \otimes V \mid \varphi(\vec{x}) = -\vec{x}\}$$

Prop: $Sym^2(V)$ and $Alt^2(V)$ are $\mathbb{C}[G]$-submods of $V \otimes V$, and $V \otimes V = Sym^2(V) \oplus Alt^2(V)$.

Cor: $\chi^2 = \chi_S + \chi_A$
where
$\chi_S(g) = \frac{1}{2}(\chi^2(g) + \chi(g^2))$ is the char of $Sym^2(V)$
$\chi_A(g) = \frac{1}{2}(\chi^2(g) - \chi(g^2))$ is the char of $Alt^2(V)$.

Representations of $G \times H$:

Let V be a $\mathbb{C}[G]$-mod and let W be a $\mathbb{C}[H]$-module. Then we can define an action of $G \times H$ on $V \otimes W$ by:

$$(g,h) \cdot (\vec{v} \otimes \vec{w}) = (g \cdot \vec{v}) \otimes (h \cdot \vec{w}).$$

Thus, $V \otimes W$ is a $\mathbb{C}[G \times H]$-module.
If V has char χ and W has char ψ, then $V \otimes W$ has char $\chi\psi$.

Thm: Let $\chi_1, \dots, \chi_k$ be the irred chars of G, and $\psi_1, \dots, \psi_\ell$ the irred chars of H. Then the irred chars of $G \times H$ are $\{\chi_i \psi_j \mid 1 \le i \le k, 1 \le j \le \ell\}$.

Example

1. $G = S_5$

g_i:	e	$(1\ 2)$	$(1\ 2)(3\ 4)$	$(1\ 2\ 3)$	$(1\ 2\ 3)(4\ 5)$	$(1\ 2\ 3\ 4)$	$(1\ 2\ 3\ 4\ 5)$
$\lvert g_i \rvert$:	1	10	15	20	20	30	24
trivial $\leftarrow \chi_1$	1	1	1	1	1	1	1
sign $\leftarrow \chi_\sigma$	1	-1	1	1	-1	-1	1
std. $\leftarrow \chi_\rho$	4	2	0	1	-1	0	-1
$\chi_{\sigma\rho}$	4	-2	0	1	1	0	-1
$(\chi_\rho)_A$	6	0	-2	0	0	0	1
$\chi_\tau = (\chi_\rho)_S - \chi_1 - \chi_\rho$	5	1	1	-1	1	-1	0
$\chi_{\sigma\tau}$	5	-1	1	-1	-1	1	0

2. $G = D_3 \times C_2 = \langle f, g \rangle \times \langle a \rangle$

g_i:	(e,e)	(e,a)	(f,e)	(f,a)	(g,e)	(g,a)
$\lvert g_i \rvert$:	1	1	2	2	3	3
$\chi_1\psi_1$	1	1	1	1	1	1
$\chi_1\psi_{-1}$	1	-1	1	-1	1	-1
$\chi_\sigma\psi_1$	1	1	1	1	-1	-1
$\chi_\sigma\psi_{-1}$	1	-1	1	-1	-1	1
$\chi_\rho\psi_1$	2	2	-1	-1	0	0
$\chi_\rho\psi_{-1}$	2	-2	-1	1	0	0

(Note that we can immediately see from this that $G/G' \cong C_2 \times C_2$ in this case.)