Example:

$G = S_4$

g_i:	e	$(1\ 2)$	$(1\ 2)(3\ 4)$	$(1\ 2\ 3)$	$(1\ 2\ 3\ 4)$
$\lvert\overline{g_i}\rvert$:	1	6	3	8	6
$\lvert Z(g_i)\rvert$:	24	4	8	3	4
χ_1	1	1	1	1	1
χ_σ	1	-1	1	1	-1
χ_ρ	3	1	-1	0	-1
$\chi_{\sigma\rho}$	3	-1	-1	0	1
χ_τ	2	0	2	-1	0

Prop: Suppose χ is a char of G and λ is a linear char. of G. Then the product $\lambda\chi$ given by $\lambda\chi(g) = \lambda(g)\chi(g)$ is also a char of G. If χ is irreducible, so is $\lambda\chi$.

Pf: Let $\rho\colon G \to GL(n,\mathbb{C})$ be a repn with char χ. Then define $\lambda\rho : G \to GL(n,\mathbb{C})$ by
$\lambda\rho(g) = \lambda(g)\rho(g)$.
Then for $g,h \in G$:

$$\lambda\rho(g)\lambda\rho(h) = \lambda(g)\rho(g)\lambda(h)\rho(h) = \lambda(g)\lambda(h)\rho(g)\rho(h)$$
$$= \lambda(gh)\rho(gh) = \lambda\rho(gh),$$

so $\lambda\rho$ is a hom. The matrix $\lambda\rho(g)$ has trace:
$tr(\lambda\rho(g)) = tr(\lambda(g)\rho(g)) = \lambda(g)\,tr(\rho(g)) = \lambda(g)\chi(g) = \lambda\chi(g)$.
Thus, $\lambda\rho$ is a repn with char $\lambda\chi$.

Now suppose χ is irreducible. Then $\langle\chi,\chi\rangle = 1$. We also know λ is irreducible, because it has degree 1, so $\langle\lambda,\lambda\rangle = \lambda\lambda^* = 1$. Therefore, $\langle\lambda\chi,\lambda\chi\rangle = \lambda\lambda^*\langle\chi,\chi\rangle = 1$, so $\lambda\chi$ is also irreducible. $\square$

Prop: Let N be a normal subgp of G, and let $\tilde{\chi}$ be a char of G/N. Let $\chi: G \to \mathbb{C}$ be given by $\chi(g) = \tilde{\chi}(Ng)$. Then χ is a char of G and χ and $\tilde{\chi}$ have the same degree.

Pf: Let $\tilde{\rho}: G/N \to GL(n,\mathbb{C})$ be a repn with char $\tilde{\chi}$. Let $\varphi: G \to G/N$ be the hom $\varphi(g) = Ng$. Then define $\rho: G \to GL(n,\mathbb{C})$ to be $\tilde{\rho}\circ\varphi$, ie $\rho(g) = \tilde{\rho}(\varphi(g)) = \tilde{\rho}(Ng)$. Thus, ρ is a repn with char χ, and ρ has the same degree as $\tilde{\rho}$. $\square$

Defn: χ is called the lift (or inflation) of $\tilde{\chi}$ to G.

Ex: $G = S_4$, $N = \{e, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\} \cong V_4$. Then $N \trianglelefteq S_4$ and $G/N \cong S_3$. ($G/N = \langle N(1\ 2\ 3), N(1\ 2)\rangle$)

Let $\tilde{\chi}$ be the deg. 2 irred char of S_3:

$\tilde{\chi}(N(1\ 2\ 3)) = -1$

$\tilde{\chi}(N(1\ 2)) = 0$.

Thus:

$\chi(e) = \tilde{\chi}(Ne) = 2$

$\chi((1\ 2)) = \tilde{\chi}(N(1\ 2)) = 0$

$\chi((1\ 2)(3\ 4)) = \tilde{\chi}(N(1\ 2)(3\ 4)) = \tilde{\chi}(Ne) = 2$

$\chi((1\ 2\ 3)) = \tilde{\chi}(N(1\ 2\ 3)) = -1$

$\chi((1\ 2\ 3\ 4)) = \tilde{\chi}(N(1\ 2\ 3\ 4)) = \tilde{\chi}(N(1\ 2)) = 0$.

Prop: Let H be a subgp of G. Then $H \trianglelefteq G$ iff H is a union of conjugacy classes of G.

Prop: If $N \trianglelefteq G$, there exist irreducible characters $\chi_1, \dots, \chi_s$ of G such that $N = \bigcap_{i=1}^{s} \ker \chi_i$, where $\ker \chi_i = \ker \rho_i = \{g \in G \mid \chi(g) = \chi(e)\}$.

Prop: G is not simple iff $\chi(g) = \chi(e)$ for some nontrivial irreducible character χ of G and for some $g \neq e$.

Defn: G' is the subgroup of G generated by elements of the form $ghg^{-1}h^{-1}$ for all $g, h \in G$. G' is called the commutator subgroup or derived subgroup of G.

Prop: Let $N \trianglelefteq G$. Then $G' \subseteq N$ iff G/N is abelian.

Prop: If χ is a linear character of G, then $G' \leq \ker \chi$.

Thm: The linear characters of G are precisely the lifts to G of the irreducible characters of G/G'

Cor: There are precisely $|G/G'|$ linear chars of G.

Example: Let $G = S_n$. Then $G' = A_n$. (To verify this, note that A_n is generated by 3-cycles, and that $(a\ b\ c) = (a\ b)(a\ b\ c)(a\ b)(a\ c\ b)$ $= (a\ b)(a\ b\ c)(a\ b)^{-1}(a\ b\ c)^{-1}$). Therefore, $G/G' \cong C_2$, so there are precisely 2 linear characters of S_n (ie the trivial character and the sign character).

Character Tables of Certain Groups of Order 12

1) $G = A_4$

Conjugacy classes:

$\bar{e} = \{e\}$

$\overline{(1\ 2)(3\ 4)} = \{(1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$

$\overline{(1\ 2\ 3)} = \{(1\ 2\ 3), (1\ 4\ 2), (1\ 3\ 4), (2\ 4\ 3)\}$

$\overline{(1\ 3\ 2)} = \{(1\ 3\ 2), (1\ 2\ 4), (1\ 4\ 3), (2\ 3\ 4)\}$

Commutator: $G' = \bar{e} \cup \overline{(1\ 2)(3\ 4)}$

Quotient: $G/G' = \{G', G'(1\ 2\ 3), G'(1\ 3\ 2)\} \cong C_3$

Char table:

g_i:	e	$(1\ 2)(3\ 4)$	$(1\ 2\ 3)$	$(1\ 3\ 2)$	
$\lvert \bar{g}_i \rvert$:	1	3	4	4	
χ_1	1	1	1	1	
χ_2	1	1	ω	ω^2	
χ_3	1	1	ω^2	ω	
χ_4	3	-1	0	0	(by orthagonality)

2) $G = D_6 = \langle f, g \mid f^6 = g^2 = e,\ fg = gf^{-1} \rangle$

Conjugacy classes: (Note that $g^{-1}fg = f^{-1}$ and $fgf^{-1} = gf^{-2}$.)

$\{e\}, \{f, f^5\}, \{f^2, f^4\}, \{f^3\}, \{g, gf^2, gf^4\}, \{gf, gf^3, gf^5\}$

Commutator: Note that $g^{-1}fgf^{-1} = f^{-2}$ and so $\overline{f^2} \subset G'$.

Also $N = \{e, f^2, f^4\}$ has order 3, so $|G/N| = 4$ and hence must be abelian. Thus, $G' = N = \bar{e} \cup \overline{f^2}$.

Quotient: $G/G' = \{G', G'f, G'g, G'gf\} \cong C_2 \times C_2$.

Char table:

g_i:	e	f	f^2	f^3	g	gf	
$\lvert \bar{g}_i \rvert$:	1	2	2	1	3	3	
χ_1	1	1	1	1	1	1	
χ_2	1	1	1	1	-1	-1	
χ_3	1	-1	1	-1	1	-1	
χ_4	1	-1	1	-1	-1	1	
χ_5	2	1	-1	-2	0	0	(tautological)
χ_6	2	-1	-1	2	0	0	($\chi_6 = \chi_3\chi_5$)

3) $G = \langle a, b \mid a^6 = e,\ a^3 = b^2,\ ab = ba^{-1} \rangle = \{ b^r a^s \mid 0 \le r \le 1, 0 \le s \le 5 \}$

Conjugacy classes: (Note that $b^{-1}ab = a^{-1}$ and $aba^{-1} = ba^{-2}$)

$\{e\}, \{a, a^5\}, \{a^2, a^4\}, \{a^3\}, \{b, ba^2, ba^4\}, \{ba, ba^3, ba^5\}$

Commutator: (As before, $b^{-1}aba^{-1} = a^{-2}$)

$G' = \bar{e} \cup \overline{a^2} = \{e, a^2, a^4\}$

Quotient: $G/G' = \{G', G'a, G'b, G'ba\} = \langle G'b \rangle = C_4$.

Char table:

g_i:	e	a	a^2	a^3	b	ba
$\lvert \bar{g_i} \rvert$	1	2	2	1	3	3
χ_1	1	1	1	1	1	1
χ_2	1	1	1	1	-1	-1
χ_3	1	-1	1	-1	i	$-i$
χ_4	1	-1	1	-1	$-i$	i
χ_5	2	1	-1	2	0	0
χ_6	2	-1	-1	-2	0	0

(by row/col. orthogonality)

Tensor Products

Defn: Let V and W be vector spaces over $\mathbb{C}$ with bases $\vec{v}_1, \ldots, \vec{v}_n$ and $\vec{w}_1, \ldots, \vec{w}_m$, respectively. Then the tensor product $V \otimes W$ is the vector space over $\mathbb{C}$ with basis given by $\{\vec{v}_i \otimes \vec{w}_j \mid 1 \leq i \leq n, 1 \leq j \leq m\}$

Defn: For $\vec{v} \in V$, $\vec{w} \in W$, we define the tensor $\vec{v} \otimes \vec{w} \in V \otimes W$ by:
$$\vec{v} \otimes \vec{w} = \left(\sum_i \lambda_i \vec{v}_i\right) \otimes \left(\sum_j \mu_j \vec{w}_j\right) = \sum_{i,j} \lambda_i \mu_j (\vec{v}_i \otimes \vec{w}_j)$$

Prop: For $\vec{v} \in V$, $\vec{w} \in W$, $\lambda \in \mathbb{C}$, we have:
$$\vec{v} \otimes (\lambda \vec{w}) = (\lambda \vec{v}) \otimes \vec{w} = \lambda(\vec{v} \otimes \vec{w}).$$

Defn: Let V, W be $\mathbb{C}[G]$-modules. Then we can define an action of G on $V \otimes W$ by:
$$g \cdot (\vec{v}_i \otimes \vec{w}_j) = (g \cdot \vec{v}_i) \otimes (g \cdot \vec{w}_j)$$

Note: For all $\vec{v} \in V$, $\vec{w} \in W$, $g \in G$, we have:
$$g \cdot (\vec{v} \otimes \vec{w}) = (g \cdot \vec{v}) \otimes (g \cdot \vec{w})$$
However, it is not true that for all $r \in \mathbb{C}[G]$:
$$r \cdot (\vec{v} \otimes \vec{w}) = (r \cdot \vec{v}) \otimes (r \cdot \vec{w}).$$

Prop: Let V and W be $\mathbb{C}[G]$-modules with characters χ and ψ, resp. Then the char of $V \otimes W$ (as a $\mathbb{C}[G]$-module) is $\chi\psi$, where $\chi\psi(g) = \chi(g)\psi(g)$.

Cor: The product of two characters of G is again a character of G.