Thm: Let χ be a character of G. Then $\langle \chi, \chi \rangle = 1$ if and only if χ is irreducible.

Pf: We have already shown that if χ is irreducible, then $\langle \chi, \chi \rangle = 1$. Now suppose χ is the char of a reducible module V. Then $V = V_1 \oplus \cdots \oplus V_r$ for irred. submods V_i. Let χ_i be the char. of V_i. Then $\chi = \chi_1 + \cdots + \chi_r$, and so
$\langle \chi, \chi \rangle = \langle \sum_{i=1}^{r} \chi_i, \sum_{i=1}^{r} \chi_i \rangle \geq r$.
Thus, $\langle \chi, \chi \rangle \neq 1$ in this case. □

The Space of Class Functions on G

Let C be the set of class functions on G. This set is a vector space under pointwise addition and scalar multiplication.

Prop: The dimension of C is equal to the number of distinct conjugacy classes of G.

Pf: Let $g_1, \dots, g_\ell$ be representatives of the distinct conjugacy classes of G. Then any function $f \in C$ is determined by its values on $g_1, \dots, g_\ell$. Let $f_i \in C$ be given by:

$$f_i(g) = \begin{cases} 1 & \text{if } g \text{ is conjugate to } g_i \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_1, \dots, f_\ell$ form a basis for C. □

Now let $\chi_1, \dots, \chi_k$ form a complete set of distinct irreducible characters of G.
(We wish to show that $\chi_1, \dots, \chi_k$ is also a basis of C.)

Prop: The chars $\chi_1, \ldots, \chi_k$ are linearly independent.

Pf: Suppose $c_1\chi_1 + \cdots + c_k\chi_k = 0$. Distinct irreducible characters form an orthonormal set. Therefore, for any i:
$\langle c_1\chi_1 + \cdots + c_k\chi_k, \chi_i \rangle = \sum_{j=1}^{k} c_j \langle \chi_j, \chi_i \rangle = c_i = 0$.
Thus, $c_i = 0 \ \forall i$, and so the χ_i are lin. indep. □

We wish to show that the χ_i also span C. We first require two preliminary facts.

Prop: Let χ be a char. of G. Then χ^* is a char. of G and χ is irred. iff χ^* is.

Pf: Let χ be the char of a repn $\rho: G \to GL(n, \mathbb{C})$. Then $\sigma: G \to GL(n, \mathbb{C})$ given by $\sigma(g) = \rho(g)^*$ is also a repn, with char χ^*. Also $\langle \chi, \chi \rangle = \langle \chi^*, \chi^* \rangle$, so $\langle \chi, \chi \rangle = 1$ iff $\langle \chi^*, \chi^* \rangle = 1$. □

Note that this implies that $\{\chi_1, \ldots, \chi_k\} = \{\chi_1^*, \ldots, \chi_k^*\}$.

Lemma: Let $f \in C$, and let $\rho: G \to GL(V)$ be an irred. repn with char χ. Then the map $\varphi: V \to V$ given by $\varphi = \sum_{g \in G} f(g)\rho(g)$ is equal to $\lambda 1_V$, where $\lambda = \frac{|G|}{n} \langle f, \chi^* \rangle$.

Pf: We observe that $\forall g \in G$:
$$\rho(g)^{-1} \varphi \rho(g) = \rho(g)^{-1} \sum_{x \in G} f(x)\rho(x)\rho(g) = \sum_{x \in G} f(x)\rho(g^{-1}xg)$$
$$= \sum_{y \in G} f(gyg^{-1})\rho(y) = \sum_{y \in G} f(y)\rho(y) = \varphi.$$
Thus, $\varphi\rho(g) = \rho(g)\varphi \ \forall g \in G$, so φ is a $\mathbb{C}[G]$-hom. By Schur's Lemma, $\varphi = \lambda 1_V$ for some $\lambda \in \mathbb{C}$, and $Tr(\varphi) = n\lambda = Tr\left(\sum_{g \in G} f(g)\rho(g)\right) = \sum_{g \in G} f(g)\chi(g) = |G| \langle f, \chi^* \rangle$. □

Prop: The chars $\chi_1, \dots, \chi_k$ span C.

PF: We will show that if $f \in C$ is such that $\langle f, \chi_i \rangle = 0 \ \forall i$, then $f = 0$. Since $\{\chi_1, \dots, \chi_k\} = \{\chi_1^*, \dots, \chi_k^*\}$, we have $\langle f, \chi_i^* \rangle = 0 \ \forall i$. Now let $\varphi_\rho = \sum_{g \in G} f(g) \rho(g)$. If ρ is irreducible, then $\varphi_\rho = 0$, by the previous lemma. If ρ is reducible, then φ_ρ acts by 0 on every irreducible subrepn, and so $\varphi_\rho = 0$ in this case also.

Now suppose ρ is the regular repn. Then for $h \in G$:

$$\varphi_\rho(h) = \sum_{g \in G} f(g) \rho(g) h = \sum_{g \in G} f(g) gh = 0.$$

Since $\{gh \mid g \in G\}$ forms a basis for the regular module, we have that $f(g) = 0 \ \forall g \in G$. Thus, $f = 0$.

Now consider any $\psi \in C$. Then let $\tilde{\psi} = \psi - \langle \psi, \chi_i \rangle \chi_i$. We can verify that $\langle \tilde{\psi}, \chi_i \rangle = 0 \ \forall i$, which implies that $\tilde{\psi} = 0$. Thus, $\psi = \langle \psi, \chi_i \rangle \chi_i$, and so $\psi \in \operatorname{span}\{\chi_1, \dots, \chi_k\}$. $\square$

Cor: $\chi_1, \dots, \chi_k$ form a basis of C.

Cor: The number of distinct irreducible characters of a group equals the number of distinct conjugacy classes.

Character Tables

Let $\chi_1, \ldots, \chi_k$ be the irreducible characters of G. Let $g_1, \ldots, g_k$ be representatives of the conjugacy classes of G.

Defn: The $k \times k$ matrix whose (i,j)-entry is $\chi_i(g_j)$ is called the character table of G.

Prop: The character table of G is an invertible matrix.

Thm: 1) Row Orthogonality:

$$\sum_{i=1}^{k} \frac{\chi_r(g_i)\chi_s(g_i^{-1})}{|Z(g_i)|} = \delta_{rs}$$

2) Column Orthogonality:

$$\sum_{i=1}^{k} \frac{\chi_i(g_r)\chi_i(g_s^{-1})}{|Z(g_s)|} = \delta_{rs}$$

Pf: 1) $$\langle \chi_r, \chi_s \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_r(g)\chi_s(g^{-1}) = \frac{1}{|G|} \sum_{i=1}^{k} \chi_r(g_i)\chi_s(g_i^{-1})|\bar{g}_i|$$

$$= \sum_{i=1}^{k} \frac{\chi_r(g_i)\chi_s(g_i^{-1})}{|Z(g_i)|} = \delta_{rs}.$$

2) Let ψ_s be such that $\psi_s(g_r) = \delta_s$. Then

$$\psi_s = \sum_{i=1}^{k} \langle \psi, \chi_i \rangle \chi_i, \text{ where } \langle \psi, \chi_i \rangle = \frac{1}{|G|} \sum_{g \in G} \psi(g)\chi_i(g^{-1})$$

But $\psi_s(g) = \begin{cases} 1 & \text{if } g \text{ is conjugate to } g_s \\ 0 & \text{otherwise} \end{cases}$.

Thus, $$\langle \psi, \chi_i \rangle = \frac{1}{|G|} |\bar{g}_s| \chi_i(g_s^{-1}) = \frac{\chi_i(g_s^{-1})}{|Z(g_s)|},$$

and so $$\psi_s(g_r) = \sum_{i=1}^{k} \frac{\chi_i(g_r)\chi_i(g_s^{-1})}{|Z(g_s)|} = \delta_{rs}.$$

$\square$

Examples of Character Tables:

1) $G = C_3$

g_i	e	a	a^2
$\lvert Z(g_i)\rvert$	3	3	3
χ_1	1	1	1
χ_2	1	ω	ω^2
χ_3	1	ω^2	ω

2) $G = D_3$

g_i	e	f	g
$\lvert Z(g_i)\rvert$	6	3	2
χ_1	1	1	1
χ_2	1	1	-1
χ_3	2	-1	0

3) $G = A_4$

g_i	g_1	g_2	g_3	g_4
$\lvert Z(g_i)\rvert$	12	4	3	3
χ_1	1	1	1	1
χ_2	1	1	ω	ω^2
χ_3	1	1	ω^2	ω
χ_4	3	-1	0	0

Prop: The regular repn of G has char χ such that:
$\chi(e) = |G|$,
$\chi(g) = 0 \quad \forall g \neq e$.

Prop: The char χ of a permutation repn is given by $\chi(g) = |\mathrm{fix}(g)|$, where $\mathrm{fix}(g)$ is the set of basis vectors fixed under the action of g.

Prop: Let χ be the char of a perm. repn. of G. Then $\tilde{\chi}(g) = |\mathrm{fix}(g)| - 1$ is also a char. of G.