Conjugacy Classes

Recall: The relation "x is conjugate to y" is an equivalence relation.

$x \sim y \iff \exists g \in G$ s.t. $x = gyg^{-1}$

The equivalence class $\bar{x}$ (or x^G) of x under this relation is the set

$\bar{x} = \{gxg^{-1} \mid g \in G\}$.

Defn: This set is called the <u>conjugacy class</u> of x in G

Examples: 1) $G = C_3 = \langle a \mid a^3 = e \rangle$

$\bar{e} = \{e\}$, $\bar{a} = \{a\}$, $\overline{a^2} = \{a^2\}$

2) $G = D_3 = \langle f, g \mid f^3 = g^2 = e, fg = gf^{-1} \rangle$

$\bar{e} = \{e\}$, $\bar{f} = \{f, f^2\}$, $\bar{g} = \{g, gf, gf^2\}$

3) $G = D_4$

$\bar{e} = \{e\}$, $\bar{f} = \{f, f^3\}$, $\overline{f^2} = \{f^2\}$, $\bar{g} = \{g, gf^2\}$, $\overline{gf} = \{gf, gf^3\}$

4) $G = S_n$

$\sigma \circ (x_1\ x_2 \cdots x_r) \circ \sigma^{-1} = (\sigma(x_1)\ \sigma(x_2)\ \cdots\ \sigma(x_3))$

$\Rightarrow$ Each conjugacy class consists of <u>all elements</u> of the same cycle shape (i.e. lengths of cycles when written as a product of disjoint cycles).

Prop: $|\bar{x}| = 1 \iff x \in Z(G)$ (where $Z(G)$ is the center of G).

Prop: Let $Z(x)$ be the centralizer of x in G, i.e. $Z(x) = \{g \in G \mid gx = xg\}$. Then $|\bar{x}| = [G : Z(x)]$.

Pf: Define $f : \bar{x} \to \{gZ(x) \mid g \in G\}$ by $f(gxg^{-1}) = gZ(x)$. We note f is surjective and observe $gxg^{-1} = hxh^{-1} \iff xg^{-1}h = g^{-1}hx \iff g^{-1}h \in Z(x) \iff gZ(x) = hZ(x)$.

So this function is well-defined and injective. □

Cor: $|\bar{x}|$ divides $|G|$.

Characters

Defn: Let $\rho : G \to GL(V)$ be a repn. of a finite group G. Then the character of ρ (or V) is the function $\chi : G \to \mathbb{C}$ given by
$\chi(g) = tr[\rho(g)]$

Recall: $tr(A+B) = tr(A) + tr(B)$
$tr(AB) = tr(BA) \Rightarrow tr(BAB^{-1}) = tr(A)$

Note: This means that the character is independent of the basis $\mathcal{B}$ of V, since if $\mathcal{B}'$ is another basis, $[\rho(g)]_{\mathcal{B}'} = T[\rho(g)]_{\mathcal{B}} T^{-1}$ for some invertible matrix T.

Prop: Let χ be the character of a repn ρ of degree n. Then for $s, t \in G$:
i) $\chi(1) = n$
ii) $\chi(s^{-1}) = \chi(s)^*$ (where if $z = x+iy$, $z^* = x-iy$).
iii) $\chi(tst^{-1}) = \chi(s)$

Remark: A function satisfying (iii) is called a class function.

Defn: We say χ is a character of G if χ is the character of some repn. of G.
We say χ is irreducible (resp reducible) if χ is the character of an irreducible (resp. reducible) repn.

Defn: A character of degree 1 is called a linear character.

Inner Products of Characters

Defn: An inner product on a vect. sp V over $\mathbb{C}$ is a function $\langle \, , \, \rangle : V \times V \to \mathbb{C}$ s.t. $\forall \vec{u}, \vec{v}, \vec{w} \in V$, $\forall \lambda \in \mathbb{C}$:

1) $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle^*$ (conjugate-symmetry)
2) $\langle \vec{u}+\vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$ } (linearity in the first argument)
3) $\langle \lambda \vec{u}, \vec{v} \rangle = \lambda \langle \vec{u}, \vec{v} \rangle$
4) $\langle \vec{u}, \vec{u} \rangle \geq 0$ and $\langle \vec{u}, \vec{u} \rangle = 0 \iff \vec{u} = \vec{0}$ } (positive-definiteness).

Defn: Let G be a finite group, and let G^* be the space of functions $G \to \mathbb{C}$. We can define an inner product on G^* by

$$\langle \phi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \phi(g)(\psi(g)^*)$$

Prop: Let G be a finite group, and let ϕ, ψ be characters of G. Then $\langle \phi, \psi \rangle$ is a real number.

Pf:

$$\langle \phi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \phi(g)(\psi(g)^*)$$

$$= \frac{1}{|G|} \sum_{g \in G} \phi(g)\psi(g^{-1}) \quad \text{(since } \chi(g^{-1}) = \chi(g)^* \text{ for chars. } \chi\text{)}$$

$$= \frac{1}{|G|} \sum_{g \in G} \phi(g^{-1})\psi(g) \quad \text{(since both sums are over all } g \in G\text{)}$$

$$= \langle \psi, \phi \rangle$$

$$= \langle \phi, \psi \rangle^* \quad \text{(by conjugate-symmetry).}$$

So $\langle \phi, \psi \rangle = \langle \phi, \psi \rangle^*$, and so $\langle \phi, \psi \rangle \in \mathbb{R}$. $\square$

Thm: Let U and V be nonisomorphic irreducible $\mathbb{C}[G]$-modules with characters χ and ψ respectively. Then:
$\langle \chi, \chi \rangle = 1$
$\langle \chi, \psi \rangle = 0$

To prove this, we first prove the following lemma

Lemma: Let U and V be irreducible $\mathbb{C}[G]$-modules corresponding to representations ρ and σ respectively, and let T be a linear map $V \to U$ given by a matrix A. Then the map $\tilde{T}: V \to U$ given by the matrix

$$\tilde{A} = \frac{1}{|G|} \sum_{g \in G} \rho(g^{-1}) A \sigma(g)$$

is a $\mathbb{C}[G]$-homorphism. Furthermore:

i) if U and V are nonisomorphic, $\tilde{A}$ is the zero matrix.

ii) if $U = V$ and $\sigma = \rho$, then $\tilde{A} = \lambda I_n$ where $\lambda = \frac{1}{n} \mathrm{Tr}(A)$.

Pf: We first show that $\tilde{T}$ is a $\mathbb{C}[G]$-hom.
We know $\tilde{T}$ is a linear (matrix) transformation. Also

$$\tilde{T}(h \cdot \vec{v}) = \frac{1}{|G|} \sum_{g \in G} \rho(g^{-1}) A \sigma(g) \sigma(h) \vec{v}$$

$$= \frac{1}{|G|} \sum_{g \in G} \rho(g^{-1}) A \sigma(gh) \vec{v}$$

$$= \frac{1}{|G|} \sum_{x \in G} \rho(hx^{-1}) A \sigma(x) \vec{v}$$

$$= \rho(h) \cdot \frac{1}{|G|} \sum_{x \in G} \rho(x^{-1}) A \sigma(x) \vec{v}$$

$$= h \cdot (\tilde{T}(\vec{v})).$$

Thus, $\tilde{T}$ is a $\mathbb{C}[G]$-hom.

By Schur's Lemma (i), we have that $\tilde{T}$ is the 0 map if $U \not\cong V$.

If $U = V$ and $\rho = \sigma$, then by Schur's Lemma (ii), we know $\tilde{T} = \lambda 1_V$ for some $\lambda \in \mathbb{C}$. Therefore, $\tilde{A} = \lambda I_n$ (where $n = \dim V$). Moreover,

$$\begin{aligned} Tr(\tilde{A}) &= n\lambda \\ &= Tr\left(\frac{1}{|G|}\sum_{g\in G} \rho(g^{-1}) A \rho(g)\right) \\ &= \frac{1}{|G|}\sum_{g\in G} Tr(\rho(g^{-1}) A \rho(g)) \\ &= \frac{1}{|G|}\sum_{g\in G} Tr(A) \\ &= Tr(A). \end{aligned}$$

Thus, $\lambda = \frac{1}{n} Tr(A)$. $\square$

Now suppose $\rho(g) = [r_{ij}(g)]$, $\sigma(g) = [s_{ij}(g)]$ and $A = [x_{ij}]$. Then for $g \in G$:

$$\rho(g^{-1}) A \sigma(g) = \sum_{j=1}^{m}\sum_{k=1}^{n} r_{ij}(g^{-1}) x_{jk} s_{kl}(g)$$

and so:

$$\tilde{A}_{il} = \frac{1}{|G|}\sum_{g\in G}\sum_{j=1}^{m}\sum_{k=1}^{n} r_{ij}(g^{-1}) x_{jk} s_{kl}(g)$$

But if $\rho \not\cong \sigma$, then $\tilde{A} = 0$ for all A. Thus,

$$\frac{1}{|G|}\sum_{g\in G} r_{ij}(g^{-1}) s_{kl}(g) = 0 \quad \forall\, i,j,k,l.$$

If $\rho = \sigma$, then $\tilde{A}_{il} = \lambda \delta_{il} = \frac{\delta_{il}}{n}\sum_{j=1}^{n} x_{jj}$, so

$$\frac{1}{|G|}\sum_{g\in G}\sum_{j=1}^{m}\sum_{k=1}^{n} r_{ij}(g^{-1}) x_{jk} r_{kl}(g) = \frac{\delta_{il}}{n}\sum_{j=1}^{n} x_{jj}$$

$$\Rightarrow \frac{1}{|G|}\sum_{g\in G} r_{ij}(g^{-1}) r_{kl}(g) = \frac{\delta_{il}\delta_{jk}}{n}.$$

We are now ready to prove our theorem.

Thm: (restated) Let U and V be nonisomorphic irreducible $\mathbb{C}[G]$-modules with characters χ and ψ respectively. Then:

$\langle \chi, \chi \rangle = 1$

$\langle \chi, \psi \rangle = 0$.

PF: Let ρ and σ be the repns corresponding to χ and ψ respectively. Then:

$$\langle \chi, \chi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\chi(g^{-1})$$

$$= \frac{1}{|G|} \sum_{g \in G} \mathrm{Tr}(\rho(g))\,\mathrm{Tr}(\rho(g^{-1}))$$

$$= \frac{1}{|G} \sum_{g \in G} \left(\sum_{i=1}^{n} r_{ii}(g) \right) \left(\sum_{j=1}^{n} r_{jj}(g^{-1}) \right)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{1}{|G|} \sum_{g \in G} r_{ii}(g)\, r_{jj}(g^{-1}) \right)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\delta_{ij}}{n}$$

$$= 1,$$

and

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\psi(g^{-1})$$

$$= \frac{1}{|G|} \sum_{g \in G} \left(\sum_{i=1}^{m} r_{ii}(g) \right) \left(\sum_{j=1}^{n} s_{jj}(g^{-1}) \right)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} \left(\frac{1}{|G|} \sum_{g \in G} r_{ii}(g)\, s_{jj}(g^{-1}) \right)$$

$$= 0.$$

$\square$

Thm: Let V be a $\mathbb{C}[G]$-module with char ψ such that $V = V_1 \oplus \cdots \oplus V_r$, where the V_i are irred. submods. Let W be an irreducible $\mathbb{C}[G]$-module with char. χ. Then $\langle \psi, \chi \rangle$ is an integer equal to the number of V_i isomorphic to W.

Pf: Let ψ_i be the characters of the V_i. Then $\psi = \psi_1 + \psi_2 + \cdots + \psi_r$. Therefore:
$$\langle \psi, \chi \rangle = \langle \sum_{i=1}^{r} \psi_i, \chi \rangle = \sum_{i=1}^{r} \langle \psi_i, \chi \rangle.$$

But since the ψ_i are irreducible chars,
$$\langle \psi_i, \chi \rangle = \begin{cases} 1 & \text{if } V_i \cong W \\ 0 & \text{if } V_i \not\cong W \end{cases}$$
Thus, $\langle \psi, \chi \rangle$ is the number of V_i isom to W. $\square$

Cor: Let U and V be $\mathbb{C}[G]$-modules with chars χ and ψ respectively. Then $U \cong V$ iff $\chi = \psi$.

Pf: If $U \cong V$, then the result follows from the fact that equivalent repns are conjugates of each other by a lin. transformation, and hence have the same trace.

If $U \not\cong V$, then we have already seen that $\chi \neq \psi$ if U and V are irreducible, since $\langle \chi, \psi \rangle = 0$ in this case. If U and V are not both irreducible, then, since $U \not\cong V$, there exists an irred module W with char ξ that appears with different multiplicity (possibly 0) in U and V. Then $\langle \chi, \xi \rangle \neq \langle \psi, \xi \rangle$, so $\chi \neq \psi$.

$\square$