Prop: Suppose G is a finite group such that every irreducible $\mathbb{C}[G]$-module has dimension 1. Then G is abelian.

Pf: We can decompose the regular module $\mathbb{C}[G]$ into a direct sum of irreducible $\mathbb{C}[G]$-submodules. Suppose $\mathbb{C}[G] = V_1 \oplus \cdots \oplus V_n$, where each V_i is irreducible. Then $\forall i$, $\dim V_i = 1$. Let $V_i = \text{span}\{\vec{v}_i\}$, so $\vec{v}_1, \dots, \vec{v}_n$ is a basis $\mathcal{B}$ for $\mathbb{C}[G]$. For all $g \in G$, $g \cdot \vec{v}_i = \lambda_i \vec{v}_i$ for some scalar $\lambda_i \in \mathbb{C}$, so

$$[g]_{\mathcal{B}} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix},$$

and in particular, $[g]_{\mathcal{B}}$ is diagonal $\forall g \in G$. Thus, for all $x, y \in G$, $[x]_{\mathcal{B}}$ and $[y]_{\mathcal{B}}$ commute (because all diagonal matrices commute). Since the map $g \mapsto [g]_{\mathcal{B}}$ is faithful, this implies that x and y commute. Hence, G is abelian. □

Decomposing the Regular $\mathbb{C}[G]$-Module:

emma: Let $T: V \to W$ be a $\mathbb{C}[G]$-hom. Then there exists a $\mathbb{C}[G]$-submod U of V such that $V = \ker(T) \oplus U$ and $U \cong \text{im}(T)$.

Pf: Define $\overline{T}: U \to \text{im}(T)$ by $\overline{T}(\vec{u}) = T(\vec{u})$. Since T is a $\mathbb{C}[G]$-hom, $\overline{T}$ is also. Now note that $\ker(\overline{T}) \subseteq U \cap \ker(T) = \{\vec{0}\}$, so $\overline{T}$ is injective. Now let $\vec{w} \in \text{im}(T)$. Then $\vec{w} = T(\vec{v})$ for some $\vec{v} \in V$ and $\vec{v} = \vec{k} + \vec{u}$ for some $\vec{k} \in \ker T$, $\vec{u} \in U$. So $\vec{w} = T(\vec{v}) = T(\vec{k}+\vec{u}) = T(\vec{k}) + T(\vec{u}) = \vec{0} + \overline{T}(\vec{u})$. Thus, $\overline{T}$ is surjective, and so $U \cong \text{im}(T)$. □

Thm: Let $V = \mathbb{C}[G]$ be the regular $\mathbb{C}[G]$-module and suppose $V = U_1 \oplus \cdots \oplus U_r$, where each U_i is an irreducible $\mathbb{C}[G]$-submodule. Then every irreducible $\mathbb{C}[G]$-module is isomorphic to one of the U_i.

Pf: Let W be an irreducible $\mathbb{C}[G]$-module, and let $\vec{w} \in W$ be nonzero. Then $\{r \cdot \vec{w} \mid r \in \mathbb{C}[G]\}$ is a $\mathbb{C}[G]$-submodule of W, and hence all of W, since W is irreducible.

Now define $T : \mathbb{C}[G] \to W$ by $T(r) = r \cdot \vec{w}$. Then T is linear and $\mathrm{im}(T) = W$. Also $T(r \cdot s) = (r \cdot s) \cdot \vec{w} = r \cdot (s\vec{w}) = r \cdot T(s)$, so T is a $\mathbb{C}[G]$-homomorphism. Then $\ker(T)$ is a submodule of $\mathbb{C}[G]$, and by Maschke's Thm, $\mathbb{C}[G] = \ker(T) \oplus U$ for some $\mathbb{C}[G]$-submod. U of $\mathbb{C}[G]$. Also $U \cong \mathrm{im}(T) = W$, and so, since W is irreducible, so is U. Thus, $U \cong U_i$ for some i, and hence $W \cong U_i$. $\square$

Cor: If G is a finite gp, then there are only finitely many non-isomorphic $\mathbb{C}[G]$-modules.

Examples: 1) $G = C_3 = \langle a \mid a^3 = e \rangle$

$\mathbb{C}[G] = \mathrm{span}\{e, a, a^2\}$

$= \mathrm{span}\{e + a + a^2\} \oplus \mathrm{span}\{e + \omega^2 a + \omega a^2\} \oplus \mathrm{span}\{e + \omega a + \omega^2 a^2\}$

2) $G = D_3 = \langle f, g \mid f^3 = g^2 = e,\ fg = gf^{-1} \rangle$

$\mathbb{C}[G] = \mathrm{span}\{e, f, f^2, g, gf, gf^2\}$

$= \mathrm{span}\{e + f + f^2 + g + gf + gf^2\} \oplus \mathrm{span}\{e + f + f^2 - g - gf - gf^2\}$

$\oplus\, \mathrm{span}\{e + \omega^2 f + \omega f^2,\ g + \omega^2 gf + \omega gf^2\} \oplus \mathrm{span}\{e + \omega f + \omega^2 f^2,\ g + \omega gf + \omega^2 gf^2\}$

(Note that the last two submods in the direct sum are isomorphic.)

The Space of $\mathbb{C}[G]$-Homs:

Defn: Let V and W be $\mathbb{C}[G]$-modules. Then $\mathrm{Hom}_{\mathbb{C}[G]}(V, W)$ is the set of $\mathbb{C}[G]$-homs from V to W. Define addition and scalar mult. of $\mathbb{C}[G]$-homs to be pointwise. Then $\mathrm{Hom}_{\mathbb{C}[G]}(V, W)$ is a vect. sp. over $\mathbb{C}$.

Note: $\mathrm{Hom}_{\mathbb{C}[G]}(V, V)$ is also referred to as $\mathrm{End}_{\mathbb{C}[G]}(V)$. This vector sp. becomes an algebra over $\mathbb{C}$ under function composition.

Prop: Suppose V, W are irreducible $\mathbb{C}[G]$-mods. Then

$$\dim(\mathrm{Hom}(V,W)) = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{if } V \not\cong W. \end{cases}$$

Pf: If $V \not\cong W$, then $\mathrm{Hom}(V,W) = \{0\}$, by Schur's Lemma. Thus, we suppose $V \cong W$. Let $\varphi : V \to W$ be a $\mathbb{C}[G]$-isom. Then if $\psi \in \mathrm{Hom}_{\mathbb{C}[G]}(V, W)$, $\varphi^{-1} \circ \psi$ is a $\mathbb{C}[G]$-isom from V to V, so by Schur's Lemma, $\varphi^{-1} \circ \psi = \lambda 1_V$ for some $\lambda \in \mathbb{C}$. Therefore, $\psi = \lambda \varphi$, so $\mathrm{Hom}_{\mathbb{C}[G]}(V, W) = \mathrm{span}\{\varphi\}$. $\square$

Defn: If V is a $\mathbb{C}[G]$-mod, and U is an irred $\mathbb{C}[G]$-mod, then we say U is a <u>composition factor</u> of V if V has a $\mathbb{C}[G]$-submod that is isomorphic to U.

Prop: Let V and W be $\mathbb{C}[G]$-mods, and suppose $\mathrm{Hom}(V, W) \neq \{0\}$. Then V and W have a common composition factor.

Pf: Let $\varphi \in \mathrm{Hom}(V, W)$ be nonzero. Then $V = \ker \varphi \oplus U$ for some submod U of V. Let X be an irred submod of U. Then $\varphi(X) \neq \{0\}$, and by Schur's Lemma, $\varphi(X) \cong X$, and $\varphi(X) \subseteq W$. $\square$

Prop: Let V, V_1, V_2, W, W_1, W_2 be $\mathbb{C}[G]$-modules. Then:

1) $\dim(\mathrm{Hom}(V, W_1 \oplus W_2)) = \dim(\mathrm{Hom}(V, W_1)) + \dim(\mathrm{Hom}(V, W_2))$

2) $\dim(\mathrm{Hom}(V_1 \oplus V_2, W)) = \dim(\mathrm{Hom}(V_1, W)) \oplus \dim(\mathrm{Hom}(V_2, W))$.

Pf: 1) We will show this by finding an invertible linear map $T: \mathrm{Hom}(V, W_1 \oplus W_2) \to \mathrm{Hom}(V, W_1) \oplus \mathrm{Hom}(V, W_2)$.

Define $\pi_i : W \to W_i$ by $\pi_1(\vec{w}_1 + \vec{w}_2) = \vec{w}_1$, $\pi_2(\vec{w}_1 + \vec{w}_2) = \vec{w}_2$ $\forall \vec{w}_1 \in W_1$ and $\vec{w}_2 \in W_2$. Then π_1, π_2 are $\mathbb{C}[G]$-homs. If $\varphi \in \mathrm{Hom}(V, W_1 \oplus W_2)$, then $\pi_i \varphi \in \mathrm{Hom}(V, W_i)$.

Now define $T: \mathrm{Hom}(V, W_1 \oplus W_2) \to \mathrm{Hom}(V, W_1) \oplus \mathrm{Hom}(V, W_2)$ by: $T(\varphi) = (\pi_1 \circ \varphi, \pi_2 \circ \varphi)$.

Then we can verify that T is linear. To see that T is invertible, we must check that it is both surjective and injective.

Let $\psi_i \in \mathrm{Hom}(V, W_i)$. Then the function

$\psi : V \to W_1 \oplus W_2$

$v \mapsto \psi_1(\vec{v}) + \psi_2(\vec{v})$

lies in $\mathrm{Hom}(V, W_1 \oplus W_2)$, and $T(\psi) = (\psi_1, \psi_2)$.

Thus, T is surjective.

Now suppose $\theta \in \ker(T)$, so $\pi_i \circ \theta(\vec{v}) = \vec{0}$ for all $\vec{v} \in V$. Then $\theta(\vec{v}) = \pi_1 \circ \theta(\vec{v}) + \pi_2 \circ \theta(\vec{v}) = \vec{0}$, so θ is the zero map, and so $\ker(T) = \{0\}$.

Thus, T is injective. $\square$

Cor: $\dim(\mathrm{Hom}(V_1 \oplus \cdots \oplus V_r, W_1 \oplus \cdots \oplus W_s)) = \sum_{i=1}^{r} \sum_{j=1}^{s} \dim(\mathrm{Hom}(V_i, W_j))$

Cor: Let V be a $\mathbb{C}[G]$-mod with $V = U_1 \oplus \cdots \oplus U_s$, where each U_i is an irred. submod. Let W be any irred. module. Then $\dim \mathrm{Hom}(V, W))$ and $\dim(\mathrm{Hom}(W, V))$ both equal the number of submods U_i such that $U_i \cong W$.

Prop: If U is a $\mathbb{C}[G]$-module, then $\dim(\mathrm{Hom}(\mathbb{C}[G], U)) = \dim U$.

Pf: Let $d = \dim U$. Choose a basis $\vec{u}_1, \ldots, \vec{u}_d$ of U. Define $\varphi_i : \mathbb{C}[G] \to U$ by $\varphi_i(r) = r \cdot \vec{u}$.
Then $\varphi_i \in \mathrm{Hom}(\mathbb{C}[G], U)$, since $\forall r, s \in \mathbb{C}[G]$
$\varphi_i(r \cdot s) = (r \cdot s) \cdot \vec{u}_i = r \cdot (s \cdot \vec{u}_i) = r \cdot \varphi_i(s)$.
We wish to show $\varphi_1, \ldots, \varphi_d$ form a basis of $\mathrm{Hom}(\mathbb{C}[G], U)$.
Let $\psi \in \mathrm{Hom}(\mathbb{C}[G], U)$. Then
$\psi(1) = \lambda_1 \vec{u}_1 + \cdots + \lambda_d \vec{u}_d$
for some $\lambda_i \in \mathbb{C}$. Since ψ is a $\mathbb{C}[G]$-hom,

$$\begin{aligned} \psi(r) &= \psi(r \cdot 1) \\ &= r \cdot \psi(1) \\ &= \lambda_1 (r \cdot \vec{u}_1) + \cdots + \lambda_d (r \cdot \vec{u}_d) \\ &= \lambda_1 \varphi_1(r) + \cdots + \lambda_d \varphi_d(r) \end{aligned}$$

Thus, the φ_i span $\mathrm{Hom}(\mathbb{C}[G], U)$.
Now suppose $\lambda_1 \varphi_1 + \cdots + \lambda_d \varphi_d = 0$ Then evaluating both sides at 1 gives us:
$\lambda_1 \vec{u}_1 + \cdots + \lambda_d \vec{u}_d = \vec{0}_U$. Since the $\vec{u}_i$ are linearly independent, we get $\lambda_i = 0 \ \forall i$, and so the φ_i are also linearly independent.
Thus, $\varphi_1, \ldots, \varphi_d$ form a basis of $\mathrm{Hom}(\mathbb{C}[G], U)$, so $\dim(\mathrm{Hom}(\mathbb{C}[G], U)) = d = \dim U$. $\square$

Thm: Suppose $\mathbb{C}[G] = U_1 \oplus \cdots \oplus U_r$, where each U_i is irred. If U is any irred. $\mathbb{C}[G]$-module, then the number of $\mathbb{C}[G]$-mods U_i with $U_i \cong U$ is $\dim U$.

Thm: Let $V_1, \ldots, V_k$ form a complete set of non-isom. irred $\mathbb{C}[G]$-modules. Then $\sum_{i=1}^{k} (\dim V_i)^2 = |G|$.