Pf of Maschke's Thm: Choose any subspace W_0 of V such that $V = U \oplus W_0$. Then $\forall \vec{v} \in V$, $\vec{v} = \vec{u} + \vec{w}$ for unique vects $\vec{u} \in U$, $\vec{w} \in W_0$. Define $\phi : V \to V$ by $\phi(\vec{v}) = \vec{u}$. Then ϕ is a projection, and $\ker \phi = W_0$, $\operatorname{im} \phi = U$.

We wish to modify ϕ to an $\mathbb{F}[G]$-hom. Let $\Psi : V \to V$ be given by

$$\Psi(\vec{v}) = \frac{1}{|G|} \sum_{g \in G} g \circ \phi \circ g^{-1} (\vec{v})$$

Then Ψ is a lin. map. such that $\operatorname{im} \Psi = U$. To verify this is an $\mathbb{F}[G]$-hom, we note $\Psi \circ h(\vec{v}) = \Psi(h \cdot \vec{v})$ (where $h \in G$).

$$= \frac{1}{|G|} \sum_{g \in G} g \circ \phi \circ g^{-1} \circ h(\vec{v})$$

$$= \frac{1}{|G|} \sum_{x \in G} h \circ x^{-1} \circ \phi \circ x (\vec{v})$$

$x = g^{-1} \circ h$
$\Rightarrow g = h x^{-1}$

$$= h \circ \Psi(\vec{v})$$

Thus, Ψ is an $\mathbb{F}[G]$-hom.

We can also check that Ψ is a projection, by verifying $\Psi(\vec{u}) = \vec{u}$ for all $\vec{u} \in U$. Let $W = \ker \Psi$. Then $V = U \oplus W$, and W is an $\mathbb{F}[G]$-submodule. □

Example: $G = \langle a \mid a^3 = e \rangle$, $V = \operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, $a \cdot \vec{v}_i = \vec{v}_{i+1 \pmod 3}$.
$U = \operatorname{span}\{\vec{v}_1 + \vec{v}_2 + \vec{v}_3\}$ is a submodule.
Let $W_0 = \operatorname{span}\{\vec{v}_1, \vec{v}_2\}$. Then $V = U \oplus W_0$ and $\phi : V \to V$ is the projection onto U:
$\phi(\vec{v}_1) = 0$, $\phi(\vec{v}_2) = 0$, $\phi(\vec{v}_1 + \vec{v}_2 + \vec{v}_3) = \vec{v}_1 + \vec{v}_2 + \vec{v}_3$
$\Psi(\vec{v}_i) = \frac{1}{3}(\vec{v}_1 + \vec{v}_2 + \vec{v}_3)$ for all $\vec{v}_i$.
$\Rightarrow W = \ker \Psi = \{\sum \lambda_i \vec{v}_i \mid \sum \lambda_i = 0\}$.

Defn: An $\mathbb{F}[G]$-module V is said to be completely reducible if $V = U_1 \oplus \cdots \oplus U_r$, where each U_i is an irreducible.

Note: V can be both irreducible and completely reducible!

Thm: If G is a group and $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$, then every nonzero $\mathbb{F}[G]$ module is completely reducible.

Pf: By induction on dim V ...

Thm: (Schur's Lemma) Let V and W be irred. $\mathbb{C}[G]$-modules.

1) If $\phi : V \to W$ is a $\mathbb{C}[G]$-hom, then either ϕ is a $\mathbb{C}[G]$-isom, or $\phi(\vec{v}) = \vec{0}_W \ \forall v \in V$.

2) If $\phi : V \to V$ is a $\mathbb{C}[G]$-isom, then ϕ is a scalar mult. of the identity map 1_V.

Pf: 1) Suppose $\phi(\vec{v}) \neq \vec{0}_W$ for some $\vec{v} \in V$. Then im $\phi \neq \{\vec{0}_W\}$. Since im ϕ is a $\mathbb{C}[G]$-submod of V, and W is irreducible, im $\phi = W$. Also, ker ϕ is a submod of V, and ker $\phi \neq V$, so ker $\phi = \{\vec{0}_V\}$ since V is irreducible. Thus, ϕ is a bijection and hence an isom.

2) The map ϕ must have an eigenvalue $\lambda \in \mathbb{C}$. Let $\psi = \phi - \lambda 1_V$, and note ker $\psi \neq \{\vec{0}_V\}$. Thus ker ψ is a nonzero $\mathbb{C}[G]$-submod of V, and hence ker $\psi = V$ since V is irred. Thus, $\psi(\vec{v}) = \vec{0} \ \forall \vec{v} \in V$, so $\phi \vec{v} = \lambda \vec{v} \ \forall v \in V$. $\square$

Prop: Let V be a nonzero $\mathbb{C}[G]$-module and suppose every $\mathbb{C}[G]$-hom from V to V is a scalar multiple of 1_V. Then V is irreducible.

Pf: Suppose V is reducible, so V has a proper nonzero submod. U. Then $V = U \oplus W$ by Maschke's Thm, and $\pi: V \to V$ s.t.
$\pi(\vec{u} + \vec{w}) = \vec{u} \quad \forall \vec{u} \in U, \vec{w} \in W$
is a $\mathbb{C}[G]$-hom which is not a scalar multiple of 1_V. $\times$

Cor: Let $\rho: G \to GL(n, \mathbb{C})$ be a repn. of G. Then ρ is irreducible iff every $n \times n$ matrix A s.t. $\rho_g A = A \rho_g \ \forall g \in G$ has the form $A = \lambda I_n$ for some $\lambda \in \mathbb{C}$.

Examples: 1) $G = C_3 = \langle a \rangle$

$\rho: G \to GL(2, \mathbb{C})$

$a \mapsto \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$.

Since this matrix commutes with all ρ_g, ρ is reducible.

2) $G = D_3 = \langle f, g \rangle$

$\rho: G \to GL(2, \mathbb{C})$

$f \mapsto \begin{bmatrix} e^{2\pi i/3} & 0 \\ 0 & e^{-2\pi i/3} \end{bmatrix}$

$g \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Suppose we have A st. $A\rho_f = \rho_f A$ and $A\rho_g = \rho_g A$.

$\omega = e^{2\pi i/3}$:

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{bmatrix} = \begin{bmatrix} a\omega & b\omega^{-1} \\ c\omega & d\omega^{-1} \end{bmatrix} \Rightarrow b = c = 0$

$\begin{bmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a\omega & b\omega \\ c\omega^{-1} & d\omega^{-1} \end{bmatrix}$

Prop: Let V be a $\mathbb{C}[G]$-module such that $V = U_1 \oplus U_2 \oplus \cdots \oplus U_n$, where the U_i are irreducible $\mathbb{C}[G]$-submods. If U is any irred. $\mathbb{C}[G]$-submod of V, then $U \cong U_k$ for some k.

Pf: For $\vec{u} \in U$, we can write $\vec{u} = \vec{u}_1 + \cdots + \vec{u}_n$ for unique vects $\vec{u}_i \in U_i$. Choose k such that $\vec{u}_k \neq \vec{0}$ for some $\vec{u} \in U$. Then the map $\pi_k : U \to U_k$ s.t. $\pi_k(\vec{u}) = \vec{u}_k$ is not the zero map.

Now note that π_k is a $\mathbb{C}[G]$-hom, and U, U_k are irreducible, so by Schur's Lemma, π_k is a $\mathbb{C}[G]$-isom. Thus, $U \cong U_k$.

Prop: If G is a finite abelian group, then every irreducible $\mathbb{C}[G]$-module has dimension 1.

Pf: Let V be an irreducible $\mathbb{C}[G]$-module. Pick $x \in G$. Since G is abelian,

$gx \cdot v = xg \cdot v \quad \forall g \in G$

Thus, the map $T_x : v \mapsto x \cdot v$ is a $\mathbb{C}[G]$-homomorphism of V. By Schur's Lemma, $T_x = \lambda_x 1_V$ for some $\lambda_x \in \mathbb{C}$.

Now, since every $x \in G$ acts on V by scalar multiplication, every subspace of V is a $\mathbb{C}[G]$-submodule. But since V is irreducible, it has no nonzero proper submodules, and hence no nonzero proper subspaces. Thus $\dim V$ must be 1. $\square$

Representation Theory of Finite Abelian Groups:

Recall: Every finite abelian group is isomorphic to a direct product of cyclic groups. (This is the Fundamental Theorem of Finite Abelian Groups.)

We first consider a cyclic group. Let $C_n = \langle a \mid a^n = e \rangle$. Let ρ be an irred repn. Then ρ has degree 1, so we have:

$$\rho: C_n \to GL(1,\mathbb{C}) \cong \mathbb{C}^\times$$
$$a \mapsto \lambda$$

Since ρ is a hom, $\rho(a^n) = \rho(e) = 1$, so $\lambda^n = 1$. Thus, λ is an n^{th} root of unity.

Now let ρ, σ be two irreducible repns. For ρ, σ to be equivalent, we require a linear isom $T: \mathbb{C} \to \mathbb{C}$ such that $T \circ \rho(x) = \sigma \circ T(x)$ $\forall x \in \mathbb{C}$.
But ρ and σ are both scalar maps, so if $\rho(a) = \lambda$ and $\sigma(a) = \mu$, we get that ρ, σ are equivalent iff $\lambda T(x) = \mu T(x)$, which occurs iff $\lambda = \mu$. Thus, we have n distinct repns (up to equivalence) of C_n, each given by mapping the generator a to a different n^{th} root of unity.

Now consider the direct product $G = C_{n_1} \times \cdots \times C_{n_r}$ of cyclic groups. Let a_i be the generator of C_{n_i}, so $o(a_i) = n_i$.
Let $g_i = (1, \ldots, 1, a_i, 1, \ldots, 1)$, with a_i in the i^{th} coordinate.
Then $g_1, \ldots, g_r$ generate G, and $o(g_i) = n_i$.

Now let ρ be an irreducible repn of G.
Then we have:

$$\rho : G \longrightarrow \mathbb{C}^\times$$
$$g_i \longmapsto \lambda_i$$

where λ_i is an n_i^{th} root of unity.
The choices for λ_i completely determine the repn, and we write the repn as $\rho_{\lambda_1, \lambda_2, \dots, \lambda_r}$.

<u>Thm:</u> Let G be the abelian gp $C_{n_1} \times \cdots \times C_{n_r}$. The repns $\rho_{\lambda_1, \dots, \lambda_r}$ of G are irreducible and have degree 1. There are $|G|$ of these repns, and every irred. repn. of G over $\mathbb{C}$ is equivalent to precisely one of these.

<u>Diagonalizability</u>

<u>Prop:</u> Let G be a fin. gp, V a $\mathbb{C}[G]$-module. If $g \in G$, then there is a basis $\mathcal{B}$ of V such that $[g]_{\mathcal{B}}$ is diagonal. If $o(g) = n$, then the diagonal entries of $[g]_{\mathcal{B}}$ are n^{th} roots of unity.

<u>Pf:</u> Let $H = \langle g \rangle$. Then $|H| = o(g) = n$. We can regard V as a $\mathbb{C}[H]$-module since H is a subgp of G. Then $V = U_1 \oplus \cdots \oplus U_k$ where the U_i are irred $\mathbb{C}[H]$-submodules and $\dim(U_i) = 1$ $\forall i$. Let $\vec{u}_i$ denote a nonzero vector in U_i. Then $\vec{u}_1, \dots, \vec{u}_k$ form a basis $\mathcal{B}$ of V, and $g \cdot \vec{u}_i = \omega_i$ $\forall i$, where ω_i is an n^{th} root of unity.
Thus, $[g]_{\mathcal{B}} = \begin{bmatrix} \omega_1 & & & 0 \\ & \omega_2 & & \\ & & \ddots & \\ 0 & & & \omega_k \end{bmatrix}$.

□