Prop: Let X be a finite G-set. Let V be the $\mathbb{F}$-span of the set $\{v_x \mid x \in X\}$. Define a multiplication $\cdot : G \times V \to V$ by
$$g \cdot \left(\sum_{x \in X} \lambda_x v_x\right) = \sum_{x \in X} \lambda_x v_{g \cdot x}.$$
Then V is an $\mathbb{F}[G]$-module (known as the <u>permutation module</u> of G over $\mathbb{F}$)

Special case: Every group G acts on itself by left multiplication, so we can form a permutation module $V = \mathrm{span}\{v_g \mid g \in G\}$, where $h \cdot v_g = v_{hg}$. The associated repn. is known as the <u>regular representation</u>.

Example: $G = S_3 = \{e, (1\ 2\ 3), (1\ 3\ 2), (1\ 2), (1\ 3), (2\ 3)\}$

1) S_3 acts on $\{1, 2, 3\}$, which gives us the permutation module $V = \mathrm{span}\{v_1, v_2, v_3\}$. This gives us a representation
$\rho : G \to GL(3, \mathbb{F})$

$$(1\ 2\ 3) \mapsto \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad (1\ 2) \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2) S_3 acts on itself by left multiplication, which gives us the regular representation:
$\rho : G \to GL(6, \mathbb{F})$

$$(1\ 2\ 3) \mapsto \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad (1\ 2) \mapsto \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The group algebra $\mathbb{F}[G]$.

G gp

$\mathbb{F}$ field

Defn: An algebra over $\mathbb{F}$ is a vector space A with an additional binary operation $\cdot : A \times A \to A$ (called multiplication) such that $\forall a, b, c \in A$, $\forall \lambda \in \mathbb{F}$:

1) $(a+b)c = ac + bc$
2) $a(b+c) = ab + ac$
3) $\lambda(ab) = (\lambda a)(b) = a(\lambda b)$

Defn: The group algebra of G over $\mathbb{F}$, denoted $\mathbb{F}[G]$, is a vector space with basis given by the elements of G, and multiplication given by

$$\left(\sum_{g \in G} \lambda_g g\right)\left(\sum_{h \in G} \mu_h h\right) = \sum_{g \in G}\sum_{h \in G} (\lambda_g \mu_h)(gh)$$

Example $G = C_3 = \{e, a, a^2\}$

$\mathbb{C}[G] = \{\lambda e + \mu a + \upsilon a^2 \mid \lambda, \mu, \upsilon \in \mathbb{C}\}$

$(3e + 2a + \frac{1}{2}a^2) \cdot (e + ia - 9a^2)$

$= 3e + 3ia - 27a^2 + 2a + 2ia^2 - 18e + \frac{1}{2}a^2 + \frac{1}{2}ie - \frac{9}{2}a$

$= (-15 + \frac{1}{2}i)e + (-\frac{5}{2} + 3i)a + (-\frac{53}{2} + 2i)a^2$

Prop: If G is a finite gp. then the vector sp $\mathbb{F}[G]$ is an $\mathbb{F}[G]$-module under the natural multiplication ($gv \in \mathbb{F}[G]$ $\forall g \in G, v \in \mathbb{F}[G]$). This is, again, the regular $\mathbb{F}[G]$-module and the associated representation is called the regular representation.

$\mathbb{F}[G]$-submodules

Defn: Let V be an $\mathbb{F}[G]$-module. A subset W of V is said to be an $\mathbb{F}[G]$-submodule of V if W is a subspace and $gw \in W$ $\forall w \in W$ and $g \in G$.

Examples: 1) If V is an $\mathbb{F}[G]$-module, then $\{0\}$ and V are both submodules.

2) Let $G = S_3$ and let $V = \text{span}\{v_1, v_2, v_3\}$ be the natural permutation module of G. Then $W = \text{span}\{v_1 + v_2 + v_3\}$ is an $\mathbb{F}[G]$-submodule.

Defn: An $\mathbb{F}[G]$-module V is said to be irreducible if it is nonzero and it has no $\mathbb{F}[G]$-submodules apart from $\{0\}$ and V. Otherwise it is said to be reducible.

Example: Let $G = D_4$ and $V = \mathbb{F}^2$, where

$$f \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -v_2 \\ v_1 \end{bmatrix}$$

$$g \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -v_1 \\ v_2 \end{bmatrix}$$

Then V is irreducible. To see this, suppose V has a nonzero proper submodule U. Then $\dim U$ must be 1, so $U = \text{span}\{\begin{bmatrix} x \\ y \end{bmatrix}\}$ for some vector $\begin{bmatrix} x \\ y \end{bmatrix}$, and $f\begin{bmatrix} x \\ y \end{bmatrix}, g \cdot \begin{bmatrix} x \\ y \end{bmatrix} \in U$. Thus,
$\begin{bmatrix} -y \\ x \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} -x \\ y \end{bmatrix} = \mu \begin{bmatrix} x \\ y \end{bmatrix}$ for some $\lambda, \mu \in \mathbb{F} - \{0\}$.

Then μ is either 1 or -1, which means either x or y is 0. Since $\lambda \neq 0$, we get that both x and y must be 0 then. Thus, $U = \{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\}$, which is a contradiction.

Defn: 1) The trivial $\mathbb{F}[G]$-module is the 1-dim vect. sp. V over $\mathbb{F}$ s.t. $gv = v \ \forall v \in V, g \in G$.

2) An $\mathbb{F}[G]$-module V is faithful if the identity elt. of G is the only elt $g \in G$ such that $gv = v \ \forall v \in V$.

$\mathbb{F}[G]$-homomorphisms

Defn: Let V and W be $\mathbb{F}[G]$-modules. Then a function $T: V \to W$ is an $\mathbb{F}[G]$-hom. if $\forall u, v \in V, \forall \lambda \in \mathbb{F}, \forall g \in G$:

1) $T(u+v) = T(u) + T(v)$
2) $T(\lambda u) = \lambda T(u)$
3) $T(g \cdot v) = g \cdot T(v)$

Defn: Let V, W be $\mathbb{F}[G]$-modules. Then $T: V \to W$ is an $\mathbb{F}[G]$-isomorphism if it is an $\mathbb{F}[G]$-hom and is invertible. In this case, we say V and W are isomorphic and write $V \cong W$.

Prop: Let V and W be $\mathbb{F}[G]$-modules and let $T: V \to W$ be an $\mathbb{F}[G]$-hom. Then $\ker(T)$ is an $\mathbb{F}[G]$-submodule of V and $\operatorname{im}(T)$ is an $\mathbb{F}[G]$-submodule of W.

Q: Is $V/\ker(T)$ an $\mathbb{F}[G]$-submodule? What does this look like?

Some Linear Algebra

Thm: If $\vec{v}_1, \ldots, \vec{v}_k$ are linearly independent vects in V, then there exist $\vec{v}_{k+1}, \ldots, \vec{v}_n$ in V s.t. $\vec{v}_1, \ldots, \vec{v}_n$ form a basis of V.

Cor: Suppose U is a subspace of V. Then $\dim U \leq \dim V$, and $\dim U = \dim V$ iff $U = V$.

Defn: If U, W are subspaces of a vect. sp. V, then $U + W = \{\vec{u} + \vec{w} \mid \vec{u} \in U, \vec{w} \in W\}$.
The sum $U + W$ is a direct sum if every elt. in the sum can be written as $\vec{u} + \vec{w}$ for $\vec{u} \in U, \vec{w} \in W$ in a unique way, and we denote this space as $U \oplus W$ in that case.

Thm: Suppose $V = U + W$, where $\vec{u}_1, \ldots, \vec{u}_k$ is a basis of U and $\vec{w}_1, \ldots, \vec{w}_\ell$ is a basis of W. Then the following are equivalent:

1) $V = U \oplus W$
2) $\vec{u}_1, \ldots, \vec{u}_k, \vec{w}_1, \ldots, \vec{w}_\ell$ is a basis of V.
3) $U \cap W = \{\vec{0}\}$.

Thm: If $\pi : V \to V$ is a projection, ie π is a linear map such that $\pi \circ \pi(\vec{v}) = \pi(\vec{v})$, then $V = \ker \pi \oplus \operatorname{im} \pi$

Direct Sums of Modules

Prop: Let U and W be $\mathbb{F}[G]$-submodules of V. Then $U \oplus W$ is also an $\mathbb{F}[G]$-module under the action $g\cdot(\vec{u}+\vec{w}) = g\cdot\vec{u} + g\cdot\vec{w}$

Prop: Suppose V is an $\mathbb{F}[G]$-module such that $V = U \oplus W$, where U, W are submodules with bases $\mathcal{B}$ and $\mathcal{C}$ respectively. Then $\mathcal{D} = \mathcal{B} \cup \mathcal{C}$ is a basis of V, and $\forall g \in G$:

$$[g]_{\mathcal{D}} = \left[\begin{array}{c|c} [g]_{\mathcal{B}} & 0 \\ \hline 0 & [g]_{\mathcal{C}} \end{array}\right]$$

Prop: Let V be an $\mathbb{F}[G]$-module and suppose $V = U_1 \oplus \cdots \oplus U_r$, where each U_i is an $\mathbb{F}[G]$-submodule of V. Then $\vec{v} \in V$ has a unique decomposition $\vec{v} = \vec{u}_1 + \cdots + \vec{u}_r$ where $\vec{u}_i \in U_i$, and we can define $\pi_i : V \to V$ by $\pi_i(\vec{v}) = \vec{u}_i$. Then each π_i is an $\mathbb{F}[G]$-hom, and a projection.

Thm: (Maschke's Thm:) Let G be a fin. gp, let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$, and let V be an $\mathbb{F}[G]$-module. If U is an $\mathbb{F}[G]$-submod of V, then there is an $\mathbb{F}[G]$-submod W of V such that $V = U \oplus W$.

nonexample: $G = C_3$, $\mathbb{F} = \mathbb{Z}_3$.

$\rho : G \to GL(2, \mathbb{F})$

$a^j \mapsto \begin{bmatrix} 1 & j \\ 0 & 1 \end{bmatrix}$

$U = \operatorname{span}\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\}$ is an $\mathbb{F}[G]$-submod, but since it is the only 1-dim submod, the desired W does not exist.