Math 375: Intro. to Representation Theory

Defn: Let F be a field. A vect. sp. over F is a set V together with operations

addition $+ : V \times V \to V$

scalar mult. $\cdot : \mathbb{F} \times V \to V$

such that:

a) V is an abelian gp under $+$

b) $\forall\, u, v \in V,\ \lambda, \mu \in \mathbb{F}$:

1) $\lambda(u+v) = \lambda u + \lambda v$

2) $(\lambda+\mu)v = \lambda v + \mu v$

3) $(\lambda\mu)v = \lambda(\mu v)$

4) $1v = v$

Defn: Let X be a set and G a gp. An action of G on X is a map $* : G \times X \to X$ s.t. $\forall\, x \in X,\ \forall\, g_1, g_2 \in G$:

1. $ex = x$
2. $g_1 g_2(x) = g_1(g_2 x)$

X is called a G-set in this case

Want: Group actions on vect. spaces.

Defn: Let V be a vect sp. over $\mathbb{F}$. Then $GL(V)$ is the gp. of isomorphisms of V onto itself.

$GL(V)$: general linear gp. of V.

Note: If $V \cong \mathbb{F}^n$ (as a vect sp), $GL(V) \cong GL(n, \mathbb{F})$ (as a gp).

Representations of Finite Groups

Defn: Let G be a group and $\mathbb{F}$ be a field (usually $\mathbb{R}$ or $\mathbb{C}$). A representation of G on a vector space V over $\mathbb{F}$ is a homomorphism ρ from G to $GL(V)$. The degree of ρ is the integer $\dim V$.

Examples: $G = D_4 = \{e, f, f^2, f^3, g, gf, gf^2, gf^3\}$
$= \langle f, g \mid f^4 = e, g^2 = e, fg = gf^{-1} \rangle$

1. $\rho : G \longrightarrow GL(2, \mathbb{F})$

$$\rho(f) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\rho(g) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

2. $\rho : G \longrightarrow GL(n, \mathbb{F})$

$\rho(x) = I_n$

(trivial repn.)

3. $\rho : G \longrightarrow GL(1, \mathbb{F}) \cong \mathbb{F}$

$\rho(f) = 1$

$\rho(g) = -1$

4. $\rho : G \rightarrow GL(4, \mathbb{F})$

$$\rho(f) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \rho(g) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Defn: Let $\rho: G \to GL(V)$ and $\sigma: G \to GL(W)$ be repns of G over $\mathbb{F}$. Then we say ρ is <u>equivalent</u> to σ if there exists an invertible linear transformation $T: V \to W$ such that:
$\sigma(g) \circ T = T \circ \rho(g) \quad \forall g \in G$.
(or equivalently, $\sigma(g) = T \circ \rho(g) \circ T^{-1}$).

$$\begin{tikzcd} V \arrow[r, "\rho(g)"] \arrow[d, "T"'] & V \arrow[d, "T"] \\ W \arrow[r, "\sigma(g)"] & W \end{tikzcd}$$

Note: Equivalence of repns is an equivalence relation.

Defn: Let ρ be a repn. of G.
i) If $\ker \rho = G$, then we say ρ is <u>trivial</u>
ii) If $\ker \rho = \{e_G\}$, then we say ρ is <u>faithful</u>.

Prop: A repn ρ of a finite gp G is faithful iff $\operatorname{im} \rho \cong G$.

PF: By the Fundamental Thm of Group Homs, if φ is a hom acting on G, $G/\ker \varphi \cong \operatorname{im} \varphi$.
($\Rightarrow$:) If ρ is faithful, $\ker \rho = \{e_G\}$ so $\operatorname{im} \rho \cong G/\ker \rho \cong G/\{e_G\} \cong G$.
($\Leftarrow$:) If $\operatorname{im} \rho \cong G$, then $G/\ker \rho \cong G$, so $|G|/|\ker \rho| = |G|$, so $|\ker \rho| = 1$, and hence $\ker \rho = \{e_G\}$. Thus, ρ must be faithful. $\square$

$\mathbb{F}[G]$-modules:

Defn: Let V be a vect. sp. over $\mathbb{F}$, and G be a gp. Then V is an $\mathbb{F}[G]$-module if there exists an operation $\cdot : G \times V \to V$ satisfying: $\forall u, v \in V$, $g, h \in G$, $\lambda \in \mathbb{F}$:

1) $e \cdot v = v$
2) $(gh) \cdot v = g \cdot (h \cdot v)$
3) $g \cdot (u+v) = g \cdot u + g \cdot v$
4) $g \cdot (\lambda v) = \lambda (g \cdot v)$

Thm: 1) If $\rho : G \to GL(V)$ is a repn of G (over $\mathbb{F}$), then V is an $\mathbb{F}[G]$-module under the multiplication given by $g \cdot v = \rho(g) v$.

2) Suppose V is an $\mathbb{F}[G]$-module and let $\mathcal{B}$ be a basis of V. Then the function $\rho : G \to GL(V)$ such that $\rho(g) = [g]_{\mathcal{B}}$ is a representation of G over $\mathbb{F}$.

Thm: Suppose that V is an $\mathbb{F}[G]$-module with basis $\mathcal{B}$, and let ρ be the representation of G over $\mathbb{F}$ defined by $\rho : g \mapsto [g]_{\mathcal{B}}$.

1) If $\mathcal{B}'$ is a basis of V, then the repn. $\phi : g \mapsto [g]_{\mathcal{B}'}$ is equivalent to ρ.

2) If σ is a repn. of G equiv. to ρ, then there is a basis $\mathcal{B}''$ of V such that: $\sigma : g \mapsto [g]_{\mathcal{B}''}$.