Math 375: Introduction to Representation Theory
Exercise Set 7

1. Suppose $\lambda, \mu \vdash n$. We say λ *dominates* μ, denoted $\lambda \trianglerighteq \mu$, if for all i,

$$\lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i.$$

   Let $\lambda > \mu$ indicate that λ follows μ in *lexicographic order*, i.e. there exists i such that $\lambda_j = \mu_j$ for all $j < i$ and $\lambda_i > \mu_i$.

   (a) Find an example of partitions $\lambda, \mu \vdash n$ such that neither $\lambda \trianglerighteq \mu$ nor $\mu \trianglerighteq \lambda$ are true. (*Hint:* Consider $n \geq 6$.)

   (b) Show that $\lambda \trianglerighteq \mu$ implies $\lambda \geq \mu$.

   (c) Show that if there exists a semistandard tableau of shape λ and content μ, then $\lambda \trianglerighteq \mu$.

2. Decompose the following modules into a direct sum of Specht modules.

   (a) $V^{(2,2,1)}$

   (b) $W^{(4,3,3,1)} \uparrow S_{12}$

3. Compute the exponential of the following matrices:

   (a) $X = \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix}$

   (b) $Y = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$

4. Find the matrices for the adjoint representation of $\mathfrak{sl}(3, \mathbb{C})$.