Math 375: Introduction to Representation Theory
Exercise Set 6

1. Let χ, ϕ and ϕ be characters of a group G. Show that $\langle \chi\psi, \phi\rangle = \langle \chi, \psi^*\phi\rangle$.

2. Suppose χ and ψ are irreducible characters of G, and ϕ is the trivial character of G. Show that
$$\langle \chi\psi, \phi\rangle = \begin{cases} 1 \text{ if } \chi = \psi^* \\ 0 \text{ if } \chi \neq \psi^* \end{cases}$$

3. There exists a group G of order 24 with precisely seven conjugacy classes with representatives $g_1, \cdots, g_7$, and has an irreducible character χ given by

g_i :	g_1	g_2	g_3	g_4	g_5	g_6	g_7
$\lvert\bar{g}_i\rvert$	1	1	6	4	4	4	4
χ	2	-2	0	$-\omega^2$	$-\omega$	ω	ω^2

   where $\omega = e^{2\pi i/3}$. Morever, we have that:
$$\begin{aligned} g_1^2, g_2^2 &\in \bar{g}_1, \\ g_3^2 &\in \bar{g}_2, \\ g_5^2, g_6^2 &\in \bar{g}_4, \\ g_4^2, g_7^2 &\in \bar{g}_5. \end{aligned}$$

   (a) Find χ_A and χ_S and show that both are irreducible.

   (b) Complete the character table of G.

4. Find the character table of $D_3 \times C_3$.

5. Let V be a $\mathbb{C}[G]$-module with character χ.

   (a) The *dual* vector space V' is the set of *linear* functions $V \to \mathbb{C}$ under pointwise addition and scalar multiplication. We can define an action of G on V' by $(g \cdot f)(v) = f(g^{-1} \cdot v)$. Show that the character of V' is χ^*.

(b) Let W be a $\mathbb{C}[G]$-module with character ψ. Let $\mathrm{Hom}(V, W)$ denote the space of linear maps from V to W (*not* necessarily $\mathbb{C}[G]$-homomorphisms). Define an action of G on $\mathrm{Hom}(V, W)$ by $(g \cdot \phi)(v) = g \cdot (\phi(g^{-1} \cdot v))$. Show that $\mathrm{Hom}(V, W)$ has character $\chi^* \psi$ by proving that it is isomorphic to $V' \otimes W$ as a $\mathbb{C}[G]$-module.

(c) Verify that $\mathrm{Hom}_{\mathbb{C}[G]}(V, W)$ consists of the set of point in $\mathrm{Hom}(V, W)$ that are fixed under the action of G specified above.

(d) Use the previous part and Exercise 1 to show that $\dim(\mathrm{Hom}_{\mathbb{C}[G]}(V, W)) = \langle \chi, \psi \rangle$.