Math 375: Introduction to Representation Theory
Exercise Set 3

1. Let V be an $\mathbb{F}[G]$-module. Show that the following maps $\varphi : V \to V$ are $\mathbb{F}[G]$-homomorphisms.

   (a) $\varphi(v) = \sum_{g \in G} g \cdot v$

   (b) $\varphi(v) = (\psi + \vartheta)(v)$, where ψ, ϑ are $\mathbb{F}[G]$-homomorphisms from V to V.

   (c) φ is the *inverse* of an $\mathbb{F}[G]$-homomorphism $\psi : V \to V$, i.e. $\varphi\psi = \psi\varphi = 1_V$

2. Let G be a finite group and let $\rho : G \to GL(2, \mathbb{C})$ be a representation of G. Suppose that there are elements g, h in G such that the matrices $\rho(g)$ and $\rho(h)$ do not commute. Prove that ρ is irreducible.

3. A *simple* group is a group that has no nontrivial, proper normal subgroups. Prove that for every finite simple group G, there exists a faithful irreducible $\mathbb{C}[G]$-module.

4. This exercise gives an example in which Maschke's Theorem fails for infinite groups. Suppose that G is the infinite group

$$\left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \middle| n \in \mathbb{Z} \right\}$$

   and let V be the $\mathbb{C}[G]$-module $\mathbb{C}^2$, with the natural matrix-vector multiplication. Show that V is not completely reducible.

5. Let $G = C_3 = \langle a \mid a^3 = e \rangle$, and let V be 3-dimensional $\mathbb{C}[G]$-module with basis v_1, v_2, v_3 such that

$$a \cdot v_i = v_{i+1 \pmod 3}.$$

   Decompose V into a direct sum of irreducible $\mathbb{C}[G]$-submodules.

6. Consider the action of $C_n = \langle x \mid x^n = e \rangle$ on $\mathbb{R}^2$ defined by letting the generator x act by rotation through ${}^{2\pi}/_{n}$.

   (a) Find an explicit representation $C_n \to GL(2, \mathbb{R})$ corresponding to this representation. Describe an explicit decomposition into irreducible subrepresentations.

   (b) Now consider the "same" action on $\mathbb{C}^2$ given by considering the target of the map described in part (a) to be the larger group $GL(2, \mathbb{C})$. Describe an explicit decomposition into irreducible sub-representations.