Math 375: Introduction to Representation Theory
Exercise Set 2

1. Let V be an $\mathbb{F}[G]$-module. Let $0, \overrightarrow{0}$ be the zero elements of $\mathbb{F}[G]$ and V, respectively.

   (a) Show that:

      i. $r \cdot \overrightarrow{0} = \overrightarrow{0}$ for all $r \in \mathbb{F}[G]$

      ii. $0 \cdot \overrightarrow{v} = \overrightarrow{0}$ for all $\overrightarrow{v} \in V$

   (b) Now let $\mathbb{F} = \mathbb{C}$ and V be the regular $\mathbb{F}[G]$-module. For $r, s \in \mathbb{C}[G]$, does $rs = 0$ necessarily imply that either $r = 0$ or $s = 0$?

2. Let V be the Klein 4-group. Write out the matrices $\rho(g)$ for each element $g \in V$, where ρ is the *regular representation* of V.

3. Prove or disprove: the regular representation is always faithful.

4. Let $V = \mathbb{R}^3$ be the natural permuation module of the symmetric group S_3.

   (a) Show that

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| \; x_1 + x_2 + x_3 = 0 \right\}$$

   is a submodule.

   (b) The representation $\rho : S_3 \to GL(W)$ is known as the *standard* representation of S_3. Fix an explicit basis of W, and find 2×2 matrices for $\rho(1\ 2\ 3)$ and $\rho(1\ 2)$ in this basis.

   (c) Let $D_3 = \{e, f, f^2, g, gf, gf^2\}$. We define the following degree 2 (*tautological*) representation of D_3:

$$\sigma(f) = \frac{1}{2} \begin{bmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$$
$$\sigma(g) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

   We can identify D_3 with S_3 by letting $f = (1\ 2\ 3)$ and $g = (1\ 2)$. Determine whether the standard representation ρ is equivalent to the tautological representation σ in this case.

5. When constucting the regular representation of G over $\mathbb{F}$, we used the fact that G can be regarded as a G-set under the action *left multiplication* of G, ie $g \cdot x = gx$ for $g, x \in G$. In this problem, we will use other actions of G on itself to construct new representations with associated module $\mathbb{F}[G]$.

   (a) Define the *right multiplication* action $\cdot : G \times G \to G$ by $g \cdot x = xg^{-1}$. Show that right multiplication is in fact an action of G on itself. Explain why the operation $g \cdot x = xg$ is not an action.

   (b) Show that the representation given by right multiplication is equivalent to the representation given by left multiplication.

   (c) Define the *conjugation* action $\cdot : G \times G \to G$ by $g \cdot x = gxg^{-1}$. Show that conjugation is in fact an action of G on itself.

   (d) Show that in general, conjugation is not equivalent to left multiplication. (Hint: Consider $G = C_3$.)