Math 375: Introduction to Representation Theory
Exercise Set 1

1. Let $D_4 = \{e, f, f^2, f^3, g, gf, gf^2, gf^3\}$. We define the following degree 2 representations of D_4:

$$\rho((g^j f^k) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}^j \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^k$$
$$\sigma((g^j f^k) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^j \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^k$$

Show that ρ and σ are equivalent representations. (Please specify an explicit linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that shows this equivalence.)

2. Give an example of a nontrivial representation of D_3 for each of the degrees 1, 2 and 3.

3. Let $C_n = \langle x \rangle$ be the cyclic group of order n generated by x. For $0 \le j < n$, let $\rho_j : C_n \to GL(1, \mathbb{C}) \cong \mathbb{C}$ be the map given by

$$\rho_j(x^t) = e^{2\pi i j t/n}.$$

   (a) For which values of j is ρ_j a representation of C_n?

   (b) For which values of j is ρ_j a faithful representation of C_n?

4. Show that equivalence of representations is an equivalence relation.

5. Suppose that ρ is a representation of G of degree 1. Prove that $G/\ker(\rho)$ is abelian.

6. Let $\phi : G \to GL(n, \mathbb{F})$ be a representation of G. Prove that the map

$$\rho : g \mapsto \det(\phi(g))$$

is also a representation of G.

7. Let $\phi : G \to GL(n, \mathbb{F})$ be a representation of G. Show that $\sigma : G/\ker(\rho) \to GL(n, \mathbb{F})$ given by $\sigma(g \ker(\rho)) = \rho(g)$ is a faithful representation of $G/\ker(\rho)$. (Note that you must show that it is a well-defined representation, as well as showing that it is faithful.)