Review Sheet

Linear Independence:

Vectors $v_1, v_2, \ldots, v_n$ are linearly independent if the only solution to the equation

$$c_1v_1 + c_2v_2 + \ldots + c_nv_n = 0$$

is $c_1 = c_2 = \ldots = c_m = 0$. Otherwise these vectors are linearly dependent.

Proofs:

- When proving linear independence: have to show the only solution to the above equation is that all coefficients equal 0
- When proving linear dependence: find an example where one vector could be expressed as the linear combo of other vectors
- When told vectors are linear independent: know that the only solution to the above equation is that all coefficients equal 0

Think:

- What does this equation look like with polynomials?

Span:

The span of vectors $v_1, v_2, \ldots, v_n$ is the set containing all vectors that could be written as the linear combinations of $v_1, v_2, \ldots, v_n$.

Proofs:

- To show whether a set of vectors $\{v_1, v_2, \ldots, v_n\}$ span a vector space W:
  (ex. Does v_1, v_2, v_3 span $\mathbb{R}^3$?)
  - Start with a vector $w \in W$ ("Let $w \in W$…", or "For any $w \in W$…")
  - Can we write $w = c_1v_1 + c_2v_2 + \ldots + c_nv_n$? (linear combo of the given vectors)
  - If so, then yes ("since this is true for all $w \in W$…span W!")
    If not, then provide a counter example (pick one specific $w \in W$)

Note:

- Sometimes the answer is obvious: "Are there enough vectors?" (e.g. 2 vectors cannot span $\mathbb{R}^3$)
- The set of vectors in a span do NOT have to be linearly independent, but in a basis they do have to be linearly independent

Basis and Dimension: connection between linear independence and span

A **basis** has to be formed by a span of linearly independent vectors. The **dimension** of the basis is the number of the vectors in the span.

Proofs:

- To prove whether some vectors form a basis for a space, prove 2 of the 3 following things (usually proving (a) and (b) is the easiest):
  a) Show that the vectors are linearly independent
  b) Count the number of vectors to make sure it matches the dimension of the space
  c) Show that the vectors span the whole space

Think:

- The basis for a line (dim=1) has 1 vector. The basis for a plane (dim=2) has 2 linearly independent vectors...
- The subspace for P_2 can never have a basis of more (or fewer) than 3 vectors

Sample Problems

Linear Independence: Chapter 2 Review #2 (p123):

Thought process:

- Need to show that for $c_1v_1 + c_2v_2 + c_3(v_1+v_2+v_3) = 0$, the only solution is $c_1= c_2= c_3=0$.
- Know that v_1, v_2, v_3 are linearly independent
- Expand expression: $c_1v_1 + c_2v_2 + c_3(v_1+v_2+v_3) = (c_1+c_3) v_1+ (c_2+c_3) v_2 + c_3v_3=0$
- Since v_1, v_2, v_3 are linearly independent…so $(c_1+c_3), (c_2+c_3), c_3$ all equal 0.
- …and reach to the conclusion that $c_1= c_2= c_3=0$.

Proof:

Consider $c_1v_1 + c_2v_2 + c_3(v_1+v_2+v_3) = 0$.

$c_1v_1 + c_2v_2 + c_3(v_1+v_2+v_3) = (c_1+c_3) v_1+ (c_2+c_3) v_2 + c_3v_3=0$

Since v_1, v_2, v_3 are linearly independent, this implies $(c_1+c_3), (c_2+c_3), c_3$ all equal 0

Therefore, $c_3=0$. $c_2+c_3= c_2+0 = 0$, so $c_2=0$. $c_1+c_3= c_1+0 = 0$, so $c_1=0$. Since the only solution for the equation is $c_1= c_2= c_3=0$, $\{v_1, v_2, v_1+v_2+v_3\}$ are linearly independent.

Span: Chapter 3 review exercise #7 (p195):

- To show that $v_2 ,\ldots v_n$ span V, need to show that any $v \in V$ can be written as the linear combo of $v_2 ,\ldots v_n$, or $v= d_2v_2 +\ldots+ d_nv_n$ for some $d_2, \ldots d_n$.

Proof:

Since $c_1\neq0$, $v_1=\frac{-c_2}{c_1}v_2 + \frac{-c_3}{c_1}v_3 + \cdots + \frac{-c_n}{c_1}v_n$. In other words, v_1 is a linear combination of $v_2 ,\ldots v_n$. We know that $v_1 ,\ldots v_n$ span V, which means that for any $v \in V$, $v= d_1v_1 +\ldots d_nv_n$ for some $d_1, \ldots d_n$. That means

$$v= d_1(\frac{-c_2}{c_1}v_2 + \frac{-c_3}{c_1}v_3 + \cdots + \frac{-c_n}{c_1}v_n) + d_2v_2 +\ldots + d_nv_n$$
$$= (d_1\frac{-c_2}{c_1}+d_2)v_2 +\ldots+ (d_1\frac{-c_n}{c_1}+d_n)v_n$$

Since v can be expressed as a linear combination of $v_2 ,\ldots v_n$ and this is true for any $v \in V$, $v_2 ,\ldots v_n$ span V and $\text{span}\{v_2 ,\ldots v_n\}=V$.

Basis: Chapter 3 review exercise #5a (p194):

- $\{v_1, v_2, v_3\}$ being a basis for v implies they are linearly independent and dim(V)=3.
- Need to show that $v_1, v_1+v_2, v_1+v_2+v_3$ are linearly independent. No need to show the set spans V because there are 3 vectors and we already know the dimension of V.

Proof:

Since V is a vector space, $v_1, v_2, v_3 \in V$ means $v_1, v_1+v_2, v_1+v_2+v_3 \in V$. Then the proof is exactly the same as in the "linear independence" problem.

After showing $v_1, v_1+v_2, v_1+v_2+v_3$ to be linearly independent and knowing that dim(V)=3, we can say that $\{v_1, v_1+v_2, v_1+v_2+v_3\}$ is a basis for V.