Math 272, Linear Algebra with Applications, Spring 2017
Final Exam Practice Test Solutions

1. Consider the matrix
$$A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 4 & -4 \\ -2 & -4 & 4 \end{bmatrix}.$$

   (a) Find all eigenvalues of A.

   (b) Find the eigenspace corresponding to each eigenvalue found in part (a).

   (c) Find a formula for A^n. Your answer should consist of a single 3×3 matrix, where the entries may depend on n.

Answer

(a) The eigenvalues of A are $0, 9$.

(b) The eigenspace corresponding 0 is
$$E = \{(-2, 1, 0)s + (2, 0, 1)r \mid r, s \in \mathbb{R}\}.$$
The eigenspace corresponding 9 is
$$E = \{(-1, -2, 2)s \mid s \in \mathbb{R}\}.$$

(c) Let
$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$
and
$$P = \begin{bmatrix} -2 & 2 & -1 \\ 1 & 0 & -2 \\ 0 & 1 & 2 \end{bmatrix}.$$
Then
$$P^{-1} = \begin{bmatrix} -(2/9) & 5/9 & 4/9 \\ 2/9 & 4/9 & 5/9 \\ -(1/9) & -(2/9) & 2/9 \end{bmatrix}$$
and
$$\begin{aligned} A^n = PD^nP^{-1} &= \begin{bmatrix} -2 & 2 & -1 \\ 1 & 0 & -2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 9^n \end{bmatrix} \begin{bmatrix} -(2/9) & 5/9 & 4/9 \\ 2/9 & 4/9 & 5/9 \\ -(1/9) & -(2/9) & 2/9 \end{bmatrix} \\ &= \begin{bmatrix} 9^{n-1} & 2 \cdot 9^{n-1} & -2 \cdot 9^{n-1} \\ 2 \cdot 9^{n-1} & 4 \cdot 9^{n-1} & -4 \cdot 9^{n-1} \\ -2 \cdot 9^{n-1} & -4 \cdot 9^{n-1} & 4 \cdot 9^{n-1} \end{bmatrix} \end{aligned}$$

2. (a) Prove that the set $S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a + d = 0 \right\}$ is a subspace of $M_{2\times 2}$, the set of 2×2 matrices.

   (b) Find a basis for S.

   (c) What is the dimension of S?

   Answer

   (a) First note $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in S$ since it satisfies the condition. Now assume $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} e & f \\ g & h \end{bmatrix} \in S$ and $\alpha \in \mathbb{R}$. Then $a + d = 0$ and $e + h = 0$. Consider

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \alpha \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a + \alpha e & b + \alpha f \\ c + \alpha g & d + \alpha h \end{bmatrix}$$

   Since $(a + \alpha e) + (d + \alpha h) = (a + d) + \alpha(e + h) = 0 + \alpha 0 = 0$, this matrix is in S, showing that S is closed under scalar multiplication and vector addition. Hence, S is a subspace of $M_{2\times 2}$.

   (b) We can see that

$$S = \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} | a, b, c \in \mathbb{R} \right\} = \left\{ a \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\}.$$

   Hence the following set spans S.

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

   Also, if $a \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, then we must have $a = b = c = 0$. Hence the set is also linearly independent. Therefore, the set given above is a basis for S.

   (c) The dimension of S is 3.

3. Let $T : V \to V$ be a linear transformation that is one to one. Show that if $\{\mathbf{v}_1, \ldots \mathbf{v}_n\}$ is a linearly independent set in V, then so is $\{T(\mathbf{v}_1), \ldots T(\mathbf{v}_n)\}$.

   Answer

   Consider the equation

$$c_1 T(\mathbf{v}_1) + \ldots + c_n T(\mathbf{v}_n) = \mathbf{0}.$$

   Since T is linear this can be rewritten as

$$T(c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n) = \mathbf{0}.$$

   Now, since T is one-to-one the kernel of T is $\{\mathbf{0}\}$, so we must have

$$c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n = \mathbf{0}.$$

Finally, since $\{\mathbf{v}_1, \dots \mathbf{v}_n\}$ is a linearly independent set we must have $c_1 = \dots = c_n = 0$. This show that the only solution to

$$c_1T(\mathbf{v}_1) + \ldots + c_nT(\mathbf{v}_n) = \mathbf{0}$$

is the trivial solution, proving that the set is linearly independent.

4. Let A be an $m \times n$ matrix and suppose $\mathbf{v}$ is a vector in null(A). Show that $\mathbf{v}$ is orthogonal to every vector in row(A). (**Hint:** Find a spanning set for row(A) and show that $\mathbf{v}$ is orthogonal to every vector in this set. Then use this to show that $\mathbf{v}$ must be orthogonal to every vector in the span of this set.)

   Answer

   The set of row vectors, $\{\mathbf{a}_1, \dots, \mathbf{a}_m\}$, of A form a spanning set for row(A). Let $\mathbf{v}$ be in null(A). Then $A\mathbf{v} = \mathbf{0}$. This implies that $\mathbf{a}_i \cdot \mathbf{v} = 0$ for each row $\mathbf{a}_i$ of A. Let $\mathbf{u}$ be any vector in row(A), since $\{\mathbf{a}_1, \dots, \mathbf{a}_m\}$ spans row(A) we can write $\mathbf{u} = c_1\mathbf{a}_1 + \dots + c_n\mathbf{a}_n$, for some $c_1, \dots, c_n \in \mathbb{R}$. Then

$$\begin{aligned}\mathbf{v} \cdot \mathbf{u} &= \mathbf{v} \cdot (c_1\mathbf{a}_1 + \dots + c_n\mathbf{a}_n) \text{ (substitution)} \\ &= c_1(\mathbf{v} \cdot \mathbf{a}_1) + \dots + c_n(\mathbf{v} \cdot \mathbf{a}_n) \text{ (properties of dot product)} \\ &= c_1 0 + \ldots c_n 0 = 0 \text{ (since } \mathbf{a}_i \cdot \mathbf{v} = 0 \text{ for each } \mathbf{a}_i).\end{aligned}$$

5. For each of the statements below, give an example of a 2×2 matrix A that satisfies the condition.

   (a) A has eigenvectors $\begin{bmatrix} 5 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 5 \end{bmatrix}$ with eigenvalues 2 and 3 respectively.

   Answer

   Let $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ and let $P = \begin{bmatrix} 5 & 4 \\ 6 & 5 \end{bmatrix}$. Then

$$A = PDP^{-1} = \begin{bmatrix} -22 & 20 \\ -30 & 27 \end{bmatrix}$$

   (b) A is the matrix representing the transformation $T : \mathcal{P}_2 \to \mathcal{P}_1$ such that $T(ax^2 + bx + c) = (3a + b)x - 2a + 4b$, relative to the bases $\mathcal{B} = \{x^2, x^2 + x, x^2 + x + 1\}$ of P_1 and $\mathcal{C} = \{x, 1\}$ of P_1.

   Answer

$$A = \begin{bmatrix} 3 & 4 & 4 \\ -2 & 2 & 2 \end{bmatrix}$$

   (c) A is a matrix such that null$(A) = \left\{ r \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ and col$(A) = \left\{ r \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

   Answer

$$A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

6. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x + y \\ x - y \end{bmatrix}.$$

(a) Show that T is an isomorphism.

Answer

We observe that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$, so we can represent T by the matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$. Since all matrix transformations are linear, we see that T must be linear in this case.

We can row reduce A to get:

$$\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \xrightarrow{1/2R_1} \begin{bmatrix} 1 & 1/2 \\ 1 & -1 \end{bmatrix} \xrightarrow{-R_1+R_2} \begin{bmatrix} 1 & 1/2 \\ 0 & -3/2 \end{bmatrix}.$$

Thus, since A has a pivot position in every column, we see that dim(ker(T))=dim(null(A))=0, so ker(T)=$\{\mathbf{0}\}$, and thus T is one-to-one. Also, dim(range(T))=dim(col(A))=2, so T is onto. Therefore, since T is one-to-one, onto and linear, T is an isomorphism.

(b) Find the inverse transformation, T^{-1}.

Answer

$$T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A^{-1}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 \\ 1/3 & -2/3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (x+y)/3 \\ (x-2y)/3 \end{bmatrix}.$$

7. Let S =Span$\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix}\right\}$.

(a) Find an orthonormal basis for S.

Answer

First, we find an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for S:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 4 \\ 4 \\ -1 \end{bmatrix} - \text{proj}_{v_1}\left(\begin{bmatrix} -1 \\ 4 \\ 4 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 4 \\ 4 \\ -1 \end{bmatrix} - \frac{6}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5/2 \\ 5/2 \\ 5/2 \\ -5/2 \end{bmatrix},$$

$$\mathbf{v}_3 = \begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix} - \text{proj}_{v_1}\left(\begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix}\right) - \text{proj}_{v_2}\left(\begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix} - \frac{4}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{10}{25}\begin{bmatrix} -5/2 \\ 5/2 \\ 5/2 \\ -5/2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 2 \\ -2 \end{bmatrix}.$$

Now we scale these to unit vectors to get an orthonormal basis:

$$\left\{ \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

(b) Find the projection of $\begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$ onto S.

Answer

$$\text{proj}_S\left(\begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}\right) = \frac{4}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{4}{4}\begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} - \frac{4}{4}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

8. Determine whether each of the following statements is true or false, give a brief justification of your answer.

(a) If A and B are $n \times n$ matrices then $\det(AB) = \det(BA)$.
True. $\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(BA)$

(b) If A is a 3×3 matrix such that $\det(A) = -2$, then $\det(3A^2) = 36$.
False. $\det(3A^2) = (3)^3(-2)^2 = 108$

(c) If the reduced row echelon form of a matrix A is the identity matrix I, then A is similar to I.
False. $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ can be reduced to I (by scaling each row by $1/2$), but A cannot be similar to I, because similar matrices have the same eigenvalues, and A has eigenvalue 2, while I only has eigenvalue 1.

(d) If A and B are similar matrices, then they have the same eigenvectors.

False. A and B have to have the same eigenvalues, but not necessarily the same eigenvectors. For example, let $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$$B = PAP^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}.$$

Then B is similar to A. However, $A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ but $B \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, so $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is an eigenvector of A, but not of B.

(e) If U is a vector space in which one can find n linearly independent vectors in U, then $\dim(U) = n$.

False. The dimension of U could also be larger than n. For example, one can find 2 linearly independent vectors in $\mathbb{R}^3$, such as $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, but $\dim(\mathbb{R}^3) > 2$.

(f) If T is a matrix transformation given by a matrix A, then dim(range(T))=dim(row(A)).

True. We know range(T)=col(A), and that dim(col(A))=rank(A)=dim(row(A). Thus, dim(range(T))=dim(row(A)).

(g) If A is an $n \times n$ matrix with at least n eigenvectors, then A is diagonalizable.

False. A must have n *linearly independent* eigenvectors in order to be diagonalizable. For instance, if $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ are both eigenvectors, but since it is not possible to find another independent eigenvector, this matrix is not diagonalizable.

(h) The transformation $T : M_{3\times3} \to M_{3\times3}$ given by $T(A) = A^t$ is linear.

True. If A and B are in $M_{3\times3}$, then

$$T(A + B) = (A + B)^t = A^t + B^t = T(A) + T(B),$$

so T preserves addition. Also, if c is a scalar, then

$$T(cA) = (cA)^t = c(A)^t = cT(A),$$

so T preserves scalar multiplication. Therefore, T is linear.