Math 272, Linear Algebra with Applications, Spring 2017
Midterm 1 Practice Test, Solutions

1. Determine all values of c for which the following linear system is consistent.

$$\begin{cases} 2x + 4y - 4z = 12 \\ 3x + 7y - 5z = 20 \\ \quad x + 3y + cz = 7 \end{cases}$$

$$\left[\begin{array}{ccc|c} 2 & 4 & -4 & 12 \\ 3 & 7 & -5 & 20 \\ 1 & 3 & c & 7 \end{array}\right] \xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 2 & -2 & 6 \\ 3 & 7 & -5 & 20 \\ 1 & 3 & c & 7 \end{array}\right]$$

$$\xrightarrow[-R_1+R_3]{-3R_1+R_2} \left[\begin{array}{ccc|c} 1 & 2 & -2 & 6 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & c+2 & 1 \end{array}\right]$$

$$\xrightarrow[-R_2+R_3]{-2R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & -4 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & c+1 & -1 \end{array}\right]$$

If $c = -1$, then the system is inconsistent. Conversely, if $c \neq -1$, then we get a pivot position in the last row of the coefficient matrix as well, so the system will be consistent.

2. Let $\mathbf{u}$ and $\mathbf{v}$ be solutions to the linear system $A\mathbf{x} = \mathbf{b}$ and let c and d be constants such that $c + d = 1$. Show that $c\mathbf{u} + d\mathbf{v}$ is also a solution to $A\mathbf{x} = \mathbf{b}$.

Since $\mathbf{u}$ and $\mathbf{v}$ are solutions to the linear system $A\mathbf{x} = \mathbf{b}$, we know that $A\mathbf{u} = \mathbf{b}$ and $A\mathbf{v} = \mathbf{b}$. Now let c and d be constants such that $c + d = 1$. Then

$$\begin{aligned} A(c\mathbf{u} + d\mathbf{v}) &= A(c\mathbf{u}) + A(d\mathbf{v}) \\ &= cA\mathbf{u} + dA\mathbf{v} \\ &= c\mathbf{b} + d\mathbf{b} \\ &= (c+d)\mathbf{b} \\ &= 1\mathbf{b} \\ &= \mathbf{b}. \end{aligned}$$

Therefore, $c\mathbf{u} + d\mathbf{v}$ is also a solution to $A\mathbf{x} = \mathbf{b}$.

3. Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 5 & \text{-4} & 2 \\ \text{-1} & 0 & 3 & \text{-1} \\ \text{-2} & 0 & 0 & 1 \end{bmatrix}$$

(a) Find $\det(A)$.

$$\begin{aligned} \det(A) &= 5 \begin{vmatrix} 1 & 0 & 1 \\ \text{-1} & 3 & \text{-1} \\ \text{-2} & 0 & 1 \end{vmatrix} && \text{(by cofactor expansion along the 2nd column)} \\ &= 5 \left(3 \begin{vmatrix} 1 & 1 \\ \text{-2} & 1 \end{vmatrix} \right) && \text{(by cofactor expansion along the 2nd column)} \\ &= 15(1 - (-2)) && \text{(by the definition of determinant for } 2 \times 2 \text{ matricies)} \\ &= 45 \end{aligned}$$

(b) Is A invertible?
A is invertible since $\det(A) \neq 0$.

4. Consider the set of vectors $\{(1, 0, 3, 1), (0, 1, -1, 1), (1, 2, 1, 0)\}$.

(a) Is the set linearly independent?

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 3 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \xrightarrow[-R_1+R_4]{-3R_1+R_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow[-R_2+R_4]{R_2+R_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{3}R_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[-2R_1+R_2]{-R_3+R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, the only solution of the equation

$$c_1(1,0,3,1) + c_2(0,1,-1,1) + c_3(1,2,1,0) = (0,0,0,0)$$

is $c_1 = c_2 = c_3 = 0$, so the set is linearly independent.

(b) Does the set span $\mathbb{R}^4$?

This set consists of only 3 vectors, and therefore cannot span a 4 dimensional space. (Also, the presence of a zero row in the reduced row echelon form of the matrix indicates that the equation $t_1(1,0,3,1) + t_2(0,1,-1,1) + t_3(1,2,1,0) = (a,b,c,d)$ could be inconsistent for some vectors (a,b,c,d).)

5. Prove that if A and B are invertible matrices, then so is A^tBA.

Let $C = A^{-1}B^{-1}(A^{-1})^t$. Then

$$\begin{aligned}
(A^tBA)C &= (A^tBA)(A^{-1}B^{-1}(A^{-1})^t) \\
&= A^tB(AA^{-1})B^{-1}(A^{-1})^t \\
&= A^t(BB^{-1})(A^{-1})^t \\
&= A^t(A^{-1})^t \\
&= (A^{-1}A)^t \\
&= I^t \\
&= I.
\end{aligned}$$

Thus, A^tBA is invertible, with inverse $A^{-1}B^{-1}(A^{-1})^t$.

6. Determine whether each of the following statements are true or false. Give a brief justification of your answer.

There are many ways to give a justification of your answer: For true statements, you might give a short argument for why it is true, state a fact that the statement follows from, or (if applicable) state that we proved it as a theorem in class or that it is proven as a theorem in the book. For false statements, you might give a specific counter example, give a general description of when it would fail i.e. what a counter example might look like, or give a correction of the statement.

(a) A homogeneous linear system of 2 equations in 3 variables will always have infinitely many solutions.

True. A 2×3 matrix necessarily has a column without a pivot position, which corresponds to a free variable, and hence infinitely many solutions.

(b) If A and B are $n \times n$ matrices then $\det(A+B) = \det(A) + \det(B)$.

False. For example,

$$\det\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right) = \det\left(\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}\right) = 4$$

but

$$\det\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right) + \det\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right) = 1 + 1 = 2.$$

(c) A set of $n+1$ vectors in $\mathbb{R}^n$ is linearly dependent.

True. The maximum number of linearly independent vectors in an n dimensional vector space is n.

(d) A set of $n-1$ vectors in $\mathbb{R}^n$ is linearly independent.

False. For example $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \right\}$ is a linearly dependent set of 2 vectors in $\mathbb{R}^3$.