Math 272, Spring 2016
Applications of Inner Products

Least Squares Approximation

Let $A\mathbf{x} = \mathbf{y}$ be an **inconsistent** system of m equations in n variables (so $\mathbf{y}$ is not in the range of A). Suppose we want to find an approximate solution to this system. One way to make this precise is to ask for a vector $\hat{\mathbf{x}}$ which minimizes $||\mathbf{y} - A\hat{\mathbf{x}}||$.

Definition. A **least squares solution** to $A\mathbf{x} = \mathbf{y}$ is an element of $\hat{\mathbf{x}} \in \mathbb{R}^n$ such that

$$||\mathbf{y} - A\hat{\mathbf{x}}|| \leq ||\mathbf{y} - A\mathbf{x}||$$

for all $\mathbf{x} \in \mathbb{R}^n$.

Suppose $\hat{\mathbf{x}}$ is a least squares solution to $A\mathbf{x} = \mathbf{y}$. Then $A\hat{\mathbf{x}}$ is the vector in range(A) which is "closest" to $\mathbf{y}$. By a theorem from last class we know that $A\hat{\mathbf{x}}$ must actually be equal to $\text{proj}_{\text{range}(A)}\,\mathbf{y}$.

$$A\hat{\mathbf{x}} = \text{proj}_{\text{range}(A)}\,\mathbf{y}$$

Computing $\hat{\mathbf{x}}$

We know that $y - \text{proj}_{\text{range}(A)}\,\mathbf{y} = \mathbf{y} - A\hat{\mathbf{x}}$ is orthogonal to every vector in range(A). Since the columns of A span range(A), we have $\mathbf{c}_i \cdot (\mathbf{y} - A\hat{\mathbf{x}}) = 0$ for each column $\mathbf{c}_i$ of A. Hence,

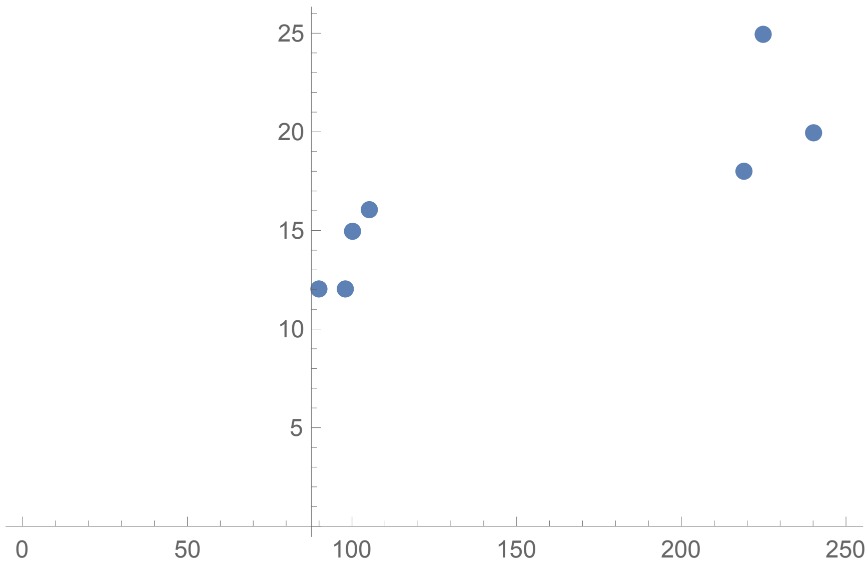
$$\begin{aligned} A^t(\mathbf{y} - A\hat{\mathbf{x}}) &= \mathbf{0} \\ A^t\mathbf{y} - A^tA\hat{\mathbf{x}} &= \mathbf{0} \\ A^tA\hat{\mathbf{x}} &= A^t\mathbf{y} \end{aligned}$$

Any solution to this equation will be a least squares solution.

Theorem. *If* rank(A) $= n$ *then* A^tA *is invertible. In this case, the above equation has exactly one solution given by* $\hat{\mathbf{x}} = (A^tA)^{-1}A^t\boldsymbol{y}$.

Example. Find a line which best fits the following data comparing the gestation period to average life span of various species: (data from 1993 World Almanac and Book of Facts)

	Gestation period in days	Longevity in years
Black bear	219	18
Grizzly bear	225	25
Polar bear	240	20
Leopard	98	12
Lion	100	15
Puma	90	12
Tiger	105	16

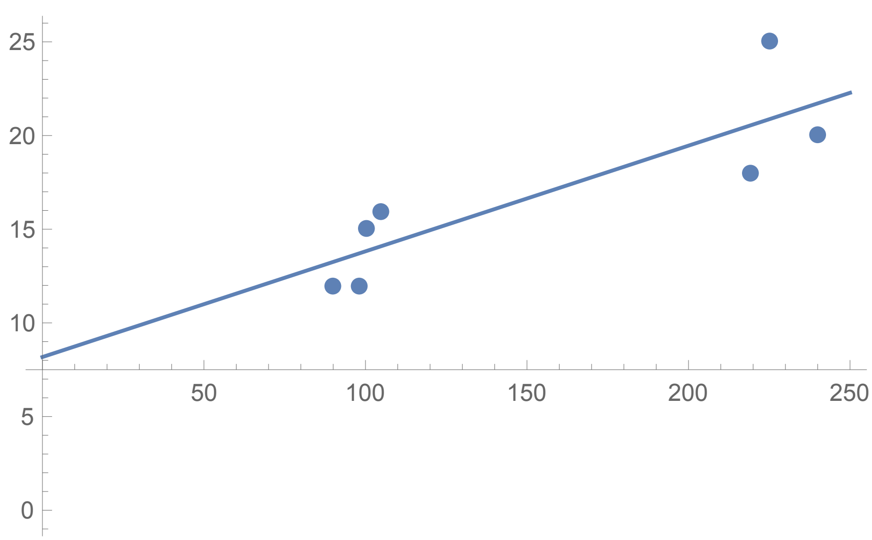


Solution. We want to find a line $y = ax + c$ which best fits this data. That is we want to find the closest possible solution to

$$\begin{bmatrix} 219 & 1 \\ 225 & 1 \\ 240 & 1 \\ 98 & 1 \\ 100 & 1 \\ 90 & 1 \\ 105 & 1 \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 18 \\ 25 \\ 20 \\ 12 \\ 15 \\ 12 \\ 16 \end{bmatrix}$$

Using Mathematica to compute the least squares solution, we find that the line of best fit is

$$y = \underline{\hspace{2cm}}\,x + \underline{\hspace{2cm}}.$$



Approximation of Functions

Definition. Let $C[a,b]$ be the vector space of continuous functions over the interval $[a,b]$. Define an inner product on $C[a,b]$ by $< f, g >= \int_a^b f(x)g(x)dx$.

Definition. Let f be an element of $C[a,b]$ and W be a subspace of $C[a,b]$. The function $g \in W$ such that $\int_a^b [f(x) - g(x)]^2 dx$ is minimal is called the least-squares approximation of f in W.

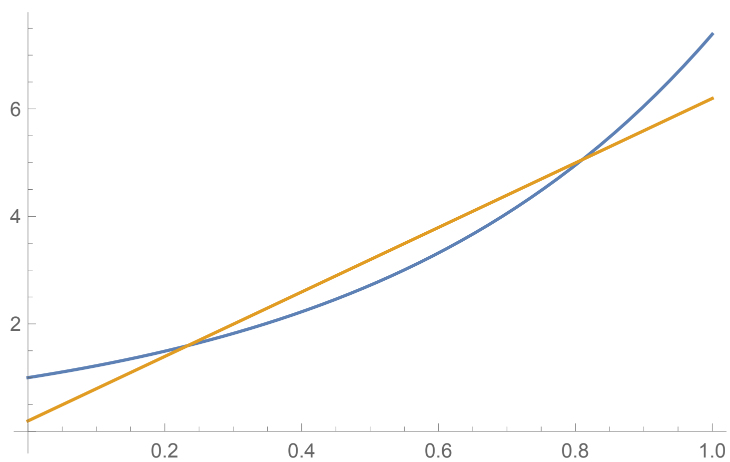
Theorem. *The least-squares approximation of f in W is given by $g = \text{proj}_W f$. If $\{u_1, \ldots, u_m\}$ is an orthonormal basis for W, this is given by*

$$g =< f, u_1 > u_1 + \cdots + < f, u_m > u_m.$$

Example. Find the least-squares linear approximation and the least squares quadratic approximation of $f(x) = e^{2x}$ over the interval $[0,1]$.

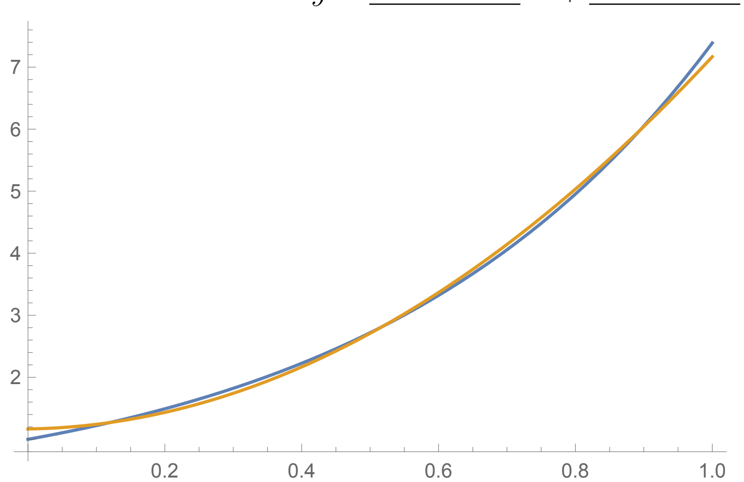
Solution. Using Mathematica and the procedure outlined above, we see the least squares line is

$$y = \_\_\_\_\_\_\_\_\_\_ x + \_\_\_\_\_\_\_\_\_\_.$$



The least squares quadratic is

$$y = \_\_\_\_\_\_\_\_\_\_ x^2 + \_\_\_\_\_\_\_\_\_\_ x + \_\_\_\_\_\_\_\_\_\_.$$



Fourier Approximation

Let $f(x)$ be a function in $C[-\pi,\pi]$ (the space of continuous functions defined on the interval $[-\pi,\pi]$). We can approximate $f(x)$ by a **trigonometric polynomial**, a function in the vector space spanned by $\{1, \cos(x), \sin(x), \cos(2x), \sin(2x), \ldots, \cos(nx), \sin(nx), \ldots\}$. Let T_n be the space spanned by $\{1, \cos(x), \sin(x), \cos(2x), \sin(2x), \ldots, \cos(nx), \sin(nx), \ldots\}$. An orthonormal basis for T_n is given by

$$\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos(x), \frac{1}{\sqrt{\pi}}\sin(x), \frac{1}{\sqrt{\pi}}\cos(2x), \frac{1}{\sqrt{\pi}}\sin(2x), \ldots, \frac{1}{\sqrt{\pi}}\cos(nx), \frac{1}{\sqrt{\pi}}\sin(nx)\}$$

Definition. The least squares approximation for a function in T_n is called the **n-th order Fourier approximation** of the function. Letting $n \to \infty$ gives the **Fourier series** of the function.

Example. Find the fourth order Fourier approximation of $f(x) = x$.

Solution. Using Mathematica and the procedure outlined above, we see that the fourth order Fourier approximation is given by

$$\begin{aligned} g(x) = \underline{\qquad\quad} &+ (\underline{\qquad\quad}\cos(x) + \underline{\qquad\quad}\sin(x)) + (\underline{\qquad\quad}\cos(2x) + \underline{\qquad\quad}\sin(2x)) \\ &+ (\underline{\qquad\quad}\cos(3x) + \underline{\qquad\quad}\sin(3x)) + (\underline{\qquad\quad}\cos(4x) + \underline{\qquad\quad}\sin(4x)). \end{aligned}$$