Conditions for a Matrix to be Invertible

Math 272, Spring 2016

THEOREM. Let A be an $n \times n$ matrix. Then the following statements are equivalent:

(1) A is invertible, i.e., there is an $n \times n$ matrix B such that $AB = BA = I_n$.

Equivalences involving the row reduction of A:

(2) A can be row reduced to I_n, i.e., the reduced echelon form of A is I_n.
(3) A is a product of elementary matrices.

Equivalences involving linear systems with A as coefficient matrix:

(4) The linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b} \in \mathbb{R}^n$.
(5) The linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution for some $\mathbf{b} \in \mathbb{R}^n$.
(6) The homogeneous linear system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Equivalences involving the columns of A:

(7) The columns of A are linearly independent in $\mathbb{R}^n$.
(8) The columns of A span $\mathbb{R}^n$.
(9) The columns of A form a basis of $\mathbb{R}^n$.

Equivalences involving matrix multiplication:

(10) There is an $n \times n$ matrix B such that $BA = I_n$.
(11) There is an $n \times n$ matrix B such that $AB = I_n$.

Other important equivalences:

(12) $\det(A) \neq 0$.
(13) $\operatorname{rank}(A) = n$.