

Math 220, Discrete Mathematics, Spring 2017

Midterm 2 Practice Test

Instructions:

- Please read each question carefully.
- No calculators, notes, books, or outside help of any kind are allowed to be used on this exam. Please turn cell phones off!
- Show all of your work and explain your answers clearly. In order to receive full credit your work must be complete, clear, and logical.
- Please cross out or fully erase any work that you do not want graded.

1. Define a relation \mathcal{R} on \mathbb{R}^2 by $(a, b)\mathcal{R}(c, d)$ iff $a \leq c$.

(a) Show that \mathcal{R} is transitive.

Suppose $(a, b)\mathcal{R}(c, d)$ and $(c, d)\mathcal{R}(e, f)$. Then $a \leq c$ and $c \leq e$, so by transitivity of \leq , we have $a \leq e$. Thus, $(a, b)\mathcal{R}(e, f)$ and so the relation \mathcal{R} is also transitive.

(b) Show that \mathcal{R} is not an equivalence relation.

This relation is not symmetric because, for example, $(1, 2)\mathcal{R}(2, 3)$ since $1 \leq 2$, but we do not have $(2, 3)\mathcal{R}(1, 2)$ since $2 \not\leq 1$.

(c) Show that \mathcal{R} is not a partial order relation.

This relation is not antisymmetric because, for example, $(1, 2)\mathcal{R}(1, 3)$ and $(1, 3)\mathcal{R}(1, 2)$ since $1 \leq 1$, but $(1, 2) \neq (1, 3)$.

2. Give an example of a partially ordered set that is not totally ordered.

Consider $\mathcal{P}(\{1, 2\})$ under the relation \subseteq . For all sets $A \in \mathcal{P}(\{1, 2\})$ we have $A \subseteq A$, so this relation is reflexive. If $A \subseteq B$ and $B \subseteq C$ then we have $A \subseteq C$ so this relation is transitive. Finally, if $A \subseteq B$ and $B \subseteq A$, then we know $A = B$ so this relation is antisymmetric. Thus, \subseteq is a partial order relation.

However, $\{1\} \not\subseteq \{2\}$ and $\{2\} \not\subseteq \{1\}$, so this relation is not a total order.

3. State and prove the divisibility test for:

(a) 11 **Claim:** Let $n = \langle d_k \dots d_1 d_0 \rangle$. Then $11|n$ if and only if $11|(-1)^k d_k + \dots - d_1 + d_0$.

Proof: We observe that since $10 \equiv (-1) \pmod{11}$, we know that $10^k \equiv (-1)^k \pmod{11}$ for all natural numbers $k \geq 2$. Therefore, $n \equiv (-1)^k d_k + \dots - d_1 + d_0 \pmod{11}$. Thus, $11|n$ if and only if $11|(-1)^k d_k + \dots - d_1 + d_0$. \square

(b) 18

Claim: Let $n = \langle d_k \dots d_1 d_0 \rangle$. Then $18|n$ if and only if $9|d_k + \dots + d_1 + d_0$ and $2|d_0$.

Proof: By the theorems proved in class, $9|n$ if and only if $9|d_k + \dots + d_1 + d_0$, and $2|n$ if and only if $2|d_0$. By a homework exercise, we know that since $\gcd(9, 2) = 1$, $18|n$ if and only if $9|n$ and $2|n$. \square

4. Define $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ by $f((a, b)) = a - b$.

(a) Determine whether f is onto.

Let r be an element of \mathbb{R} . Then $(r, 0)$ is in the set \mathbb{R}^2 and $f((r, 0)) = r - 0 = r$, so the set f is onto.

(b) Find $f^{-1}(\{0\})$.

$$f^{-1}(\{0\}) = \{(a, a) \mid a \in \mathbb{R}\}$$

(c) Determine whether f is invertible.

f is not invertible since it is not 1-1. For instance, $f((1, 1)) = f((2, 2)) = 0$.

5. Prove the following statement or give a counterexample: Let $a, b, c \in \mathbb{N}$. If $\gcd(a, b) \neq 1$ and $a \mid (b + c)$ then $\gcd(a, c) \neq 1$.

This statement is **true**. Since $a \mid (b + c)$ there is some $n \in \mathbb{Z}$ such that $b + c = an$, and so $c = an - b$. Let $g = \gcd(a, b)$. Then $g \mid a$ and $g \mid b$, and so we must have $g \mid (an - b)$ and thus $g \mid c$. This implies that g is a common divisor of a and c and hence $g \leq \gcd(a, c)$. Since $g > 1$, we must have $\gcd(a, c) > 1$ also.

6. Compute the following:

(a) The number of ways of ordering 3 *different* one topping pizzas, given that there is a choice of 10 possible toppings.

$$10 \cdot 9 \cdot 8$$

[Note: I would also accept treating this as an unordered arrangement, which would give you $10 \cdot 9 \cdot 8 / 3!$ ways. I will attempt to disambiguate more carefully on the exam to avoid this confusion.]

(b) The number of different license plates if each contains 1 number followed by two letters followed by 3 numbers.

$$10 \cdot 26^2 \cdot 10^3$$

(c) The number of positive divisors of $2^8 3^2 5^7 11^3$.

$$9 \cdot 3 \cdot 8 \cdot 4$$

(d) The number of cards to be dealt from a standard deck to guarantee 5 cards of the same suit.

$$4(4) + 1$$