Math 220, Discrete Mathematics, Spring 2017
Midterm 1 Practice Test Solutions

Instructions:

- **Please read each question carefully.**
- **No calculators, notes, books, or outside help of any kind are allowed to be used on this exam. Please turn cell phones off!**
- **Show all of your work and explain your answers clearly. In order to receive full credit your work must be complete, clear, and logical.**
- **Please cross out or fully erase any work that you do not want graded.**

1. Let a be an integer. Prove $a^2 - 2a + 5$ is even if and only if a is odd.

   ($\Leftarrow$:) Suppose a is odd. Then $a = 2k + 1$ for some $k \in \mathbb{Z}$, and so

$$\begin{aligned} a^2 - 2a + 5 &= (2k+1)^2 - 2(2k+1) + 5 \\ &= 4k^2 + 4k + 1 - 4k - 2 + 5 \\ &= 4k^2 + 4 \\ &= 2(2k^2 + 2). \end{aligned}$$

   Thus, $a^2 - 2a + 5$ is even.

   ($\Rightarrow$:) Now suppose a is even. Then $a = 2m$ for some $m \in \mathbb{Z}$, and so

$$\begin{aligned} a^2 - 2a + 5 &= (2m)^2 - 2(2m) + 5 \\ &= 4m^2 - 4m + 5 \\ &= 2(2m^2 - 2m + 2) + 1. \end{aligned}$$

   Thus $a^2 - 2a + 5$ is odd in this case, and therefore $a^2 - 2a + 5$ is even if and only if a is odd.

2. Prove

$$-1 + 2 - 3 + \cdots - (2n-1) + 2n = n$$

   for all natural numbers n.

   We use induction on n.

   Base case: Let $n = 1$. Then $-1 + 2 = 1$.

   Inductive step: Suppose $-1 + 2 - 3 + \cdots - (2k-1) + 2k = k$ for some $k \in \mathbb{N}$. Then

$$\begin{aligned} &-1 + 2 - 3 + \cdots - (2k-1) + 2k - (2(k+1) - 1) + 2(k+1) \\ &\quad = k - (2(k+1) - 1) + 2(k+1) \text{ (by the inductive hypothesis)} \\ &\quad = k - 2k - 1 + 2k + 2 \\ &\quad = k + 1. \end{aligned}$$

   Therefore, the statement holds for all $n \in \mathbb{N}$.

3. Let D_k be the set of *prime* divisors of k.

   (a) Find D_1 and D_{20}.

   (b) Find $|D_{10} \times D_{15}|$.

   (c) Find the power set of D_6.

   a. $D_1 = \varnothing, D_{20} = \{2, 5\}$.

   b. $D_{10} = \{2, 5\}, D_{15} = \{3, 5\}$, so $|D_{10} \times D_{15}| = |D_{10}||D_{15}| = 2 \cdot 2 = 4$.

   c.$D_6 = \{2, 3\}$ so $\mathcal{P}(D_6) = \{\varnothing, \{2\}, \{3\}, \{2, 3\}\}$.

4. Use the Euclidean Algorithm to find gcd(1320, 231)

$$\begin{aligned} 1320 &= 5(231) + 165 \\ 231 &= 1(165) + 66 \\ 165 &= 2(66) + 33 \\ 66 &= 2(33) + 0. \end{aligned}$$

   Thus $\gcd(1320, 231) = 33$.

5. Construct a truth table to show that the statement $P \Rightarrow Q$ is equivalent to the statement $(\sim P) \vee Q$.

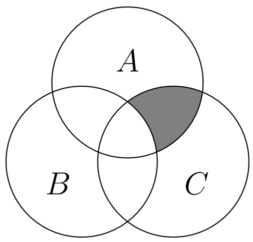
P	Q	$P \Rightarrow Q$	$\sim P$	$(\sim P) \vee Q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

6. Suppose A, B, and C are sets. Explain whether the following are true or false.

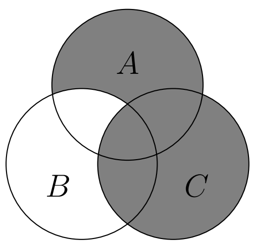
   (a) $(A \backslash B) \cap C = (C \backslash B) \cap A$.

   (b) $(A \backslash B) \cup C = (C \backslash B) \cup A$.

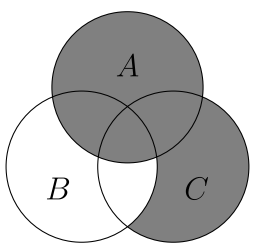
   a. True. Both regions are given by:



   b. False. $(A \backslash B) \cup C$ is given by:



   whereas $(C \backslash B) \cup A$ is given by:



7. Consider the statement: $\forall a \in \mathbb{Z}$ and $b \in \mathbb{N}$, $\exists c \in \mathbb{N}$ such that $ac > ab$.

   (a) Write down the negation of the statement.

   (b) Is the original statement true or false? Explain your answer.

   a. $\exists a \in \mathbb{Z}$ and $b \in \mathbb{N}$ such that $\forall c \in \mathbb{N}$, $ac \leq ab$.

   b. The original statement is false. If $a = 0, b = 1$, then $ac = ab = 0$ for all $c \in \mathbb{N}$, and so there does not exist $c \in \mathbb{N}$ such that $ac > ab$.