Math 220, Discrete Mathematics, Spring 2017
Midterm 2 Practice Test

Instructions:

- **Please read each question carefully.**
- **No calculators, notes, books, or outside help of any kind are allowed to be used on this exam. Please turn cell phones off!**
- **Show all of your work and explain your answers clearly. In order to receive full credit your work must be complete, clear, and logical.**
- **Please cross out or fully erase any work that you do not want graded.**

1. Define a relation $\mathcal{R}$ on $\mathbb{R}^2$ by $(a,b)\mathcal{R}(c,d)$ iff $a \leq c$.

   (a) Show that $\mathcal{R}$ is transitive.

   Suppose $(a,b)\mathcal{R}(c,d)$ and $(c,d)\mathcal{R}(e,f)$. Then $a \leq c$ and $c \leq e$, so by transitivity of $\leq$, we have $a \leq e$. Thus, $(a,b)\mathcal{R}(e,f)$ and so the relation $\mathcal{R}$ is also transitive.

   (b) Show that $\mathcal{R}$ is not an equivalence relation.

   This relation is not symmetric because, for example, $(1,2)\mathcal{R}(2,3)$ since $1 \leq 2$, but we do not have $(2,3)\mathcal{R}(1,2)$ since $2 \not\leq 1$.

   (c) Show that $\mathcal{R}$ is not a partial order relation.

   This relation is not antisymmetric because, for example, $(1,2)\mathcal{R}(1,3)$ and $(1,3)\mathcal{R}(1,2)$ since $1 \leq 1$, but $(1,2) \neq (1,3)$.

2. Give an example of a partially ordered set that is not totally ordered.

   Consider $\mathcal{P}(\{1,2\})$ under the relation $\subseteq$. For all sets $A \in \mathcal{P}(\{1,2\})$ we have $A \subseteq A$, so this relation is reflexive. If $A \subseteq B$ and $B \subseteq C$ then we have $A \subseteq C$ so this relation is transitive. Finally, if $A \subseteq B$ and $B \subseteq A$, then we know $A = B$ so this relation is antisymmetric. Thus, $\subseteq$ is a partial order relation.

   However, $\{1\} \not\subseteq \{2\}$ and $\{1\} \not\subseteq \{2\}$, so this relation is not a total order.

3. State and prove the divisibility test for:

   (a) 11 **Claim:** Let $n = \langle d_k \dots d_1 d_0 \rangle$. Then $11|n$ if and only if $11|(-1)^k d_k + \cdots - d_1 + d_0$.

   Proof: We observe that since $10 \equiv (-1) \mod 11$, we know that $10^k \equiv (-1)^k \mod 11$ for all natural numbers $k \geq 2$. Therefore, $n \equiv (-1)^k d_k + \cdots - d_1 + d_0 \mod 11$. Thus, $11|n$ if an only if $11|(-1)^k d_k + \cdots - d_1 + d_0$. □

   (b) 18

   Claim: Let $n = \langle d_k \dots d_1 d_0 \rangle$. Then $18|n$ if and only if $9|d_k + \cdots + d_1 + d_0$ and $2|d_0$.

   Proof: By the theorems proved in class, $9|n$ if and only if $9|d_k + \cdots + d_1 + d_0$, and $2|n$ if and only if $2|d_0$. By a homework exercise, we know that since $\gcd(9,2) = 1$, $18|n$ if and only if $9|n$ and $2|n$. □

4. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f((a,b)) = a - b$.
   (a) Determine whether f is onto.
   Let r be an element of $\mathbb{R}$. Then $(r, 0)$ is in the set $\mathbb{R}^2$ and $f((r,0)) = r - 0 = r$, so the set f is onto.
   (b) Find $f^{-1}(\{0\})$.
   $f^{-1}(\{0\}) = \{(a,a) \mid a \in \mathbb{R}\}$
   (c) Determine whether f is invertible.
   f is not invertible since it is not 1-1. For instance, $f((1,1)) = f((2,2)) = 0$.

5. Prove the following statement or give a counterexample: Let $a, b, c \in \mathbb{N}$. If $\gcd(a,b) \neq 1$ and $a|(b+c)$ then $\gcd(a,c) \neq 1$.

   This statement is **true**. Since $a|(b+c)$ there is some $n \in \mathbb{Z}$ such that $b + c = an$, and so $c = b - an$. Let $g = \gcd(a,b)$. Then $g|b$ and $g|a$, and so we must have $g|(b - an)$ and thus $g|c$. This implies that $g|\gcd(a,c)$ and since $g > 1$, we must have $\gcd(a,c) > 1$ also.

6. Compute the following:
   (a) The number of ways of ordering 3 *different* one topping pizzas, given that there is a choice of 10 possible toppings.
   $10 \cdot 9 \cdot 8$
   (b) The number of different license plates if each contains 1 number followed by two letters followed by 3 numbers.
   $10 \cdot 26^2 \cdot 10^3$
   (c) The number of positive divisors of $2^8 3^2 5^7 11^3$.
   $9 \cdot 3 \cdot 8 \cdot 4$
   (d) The number of cards to be dealt from a standard deck to guarantee 5 cards of the same suit.
   $4(4) + 1$