Analytic and Asymptotic Methods for Nonlinear Singularity Analysis: a Review and Extensions of Tests for the Painlevé Property

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Abstract

The integrability (solvability via an associated single-valued linear problem) of a differential equation is closely related to the singularity structure of its solutions. In particular, there is strong evidence that all integrable equations have the Painlevé property, that is, all solutions are single-valued around all movable singularities. In this expository article, we review methods for analysing such singularity structure. In particular, we describe well known techniques of nonlinear regular-singular-type analysis, i.e. the Painlevé tests for ordinary and partial differential equations. Then we discuss methods of obtaining sufficiency conditions for the Painlevé property. Recently, extensions of *irregular* singularity analysis to nonlinear equations have been achieved. Also, new asymptotic limits of differential equations preserving the Painlevé property have been found. We discuss these also.

1 Introduction

A differential equation is said to be integrable if it is solvable (for a sufficiently large class of initial data) via an associated (single-valued) linear problem.

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A famous example is the Korteweg-de Vries equation (KdV),

$$u_t + 6uu_x + u_{xxx} = 0, \tag{1.1}$$

where the subscripts denote partial differentiation.

The KdV equation was discovered to be integrable by Gardner, Greene, Kruskal, and Miura [21]. (Its method of solution is called the *inverse scattering transform* (IST) method; see the paper by Mark Ablowitz in the present collection.) Since this discovery, a large collection of nonlinear equations (see [1]) has been identified to be integrable. These range over many dimensions and include not just partial differential equations (PDEs) but also differential-difference equations, integro-differential equations, and ordinary differential equations (ODEs).

Six classical nonlinear second-order ODEs called the Painlevé equations are prototypical examples of integrable ODEs. They possess a characteristic singularity structure i.e. all movable singularities of all solutions are poles. Movable here means that the singularity's position varies as a function of initial values. A differential equation is said to have the Painlevé property if all solutions are single-valued around all movable singularities. (See comments below and in Section 2 on variations of this definition.) Thefore, the Painlevé equations possess the Painlevé property. Painlevé [55], Gambier [20], and R. Fuchs [19] identified these equations (under some mild conditions) as the only ones (of second order and first degree) with the Painlevé property whose general solutions are new transcendental functions.

Integrable equations are rare. Perturbation of such equations generally destroys their integrability. On the other hand, any constructive method of identifying the integrability of a given system contains severe shortcomings. The problem is that if a suitable associated linear problem cannot be found it is unclear whether the fault lies with the lack of integrability of the nonlinear system or with the lack of ingenuity of the investigator. So the identification of integrability has come to rely on other evidence, such as numerical studies and the singularity structure of the system.

There is strong evidence [60, 61] that the integrability of a nonlinear sytem is intimately related to the singularity structure admitted by the system in its solutions. Dense multi-valuedness (branching) around movable singularities of solutions is an indicator of nonintegrability[62]. The Painlevé property excludes such branching and has been proposed as a pointer to integrability.

The complex singularity structure of solutions was first used by Kowalevskaya [39, 42] to identify an integrable case of the equations of motion for a rotating top. Eighty eight years later, this connection was reobserved in the context of integrable PDEs by Ablowitz and Segur [5], and Ablowitz, Ramani, and Segur [3, 4]. Their observations led to the following conjecture.

The ARS Conjecture: *Any ODE which arises as a reduction of an integrable PDE possesses the Painlevé property, possibly after a transformation of variables.*

For example, the sine-Gordon equation

$$u_{xt} = \sin u, \tag{1.2}$$

which is well known to be integrable [2, 1], admits the simple scaling symmetry

$$x \to \lambda x, \quad t \to \lambda^{-1} t. \tag{1.3}$$

To find a reduction with respect to the symmetry (1.3), restrict to the subspace of solutions that is invariant under (1.3) by introducing new variables z, w such that

$$u(x,t) = w(z), \quad z = xt.$$

This gives

$$zw'' + w' = \sin w, \tag{1.4}$$

where the prime denotes differentiation with respect to z. To investigate the Painlevé property, this equation must first be transformed to one that is rational (or possibly algebraic) in w. (Otherwise, the nonlinear analogue of Fröbenius analysis used to investigate the Painlevé property cannot find a leading-order term to get started. See Section 2.) Introduce the new dependent variable $y := \exp(iw)$. Then equation (1.4) becomes

$$z(yy'' - y'^2) + yy' = \frac{1}{2}y(y^2 - 1).$$

This equation (a special case of the third Painlevé equation) can be shown to have the Painlevé property. (See Section 2.)

There is now an overwhelming body of evidence for the ARS conjecture. A version that is directly applicable to PDEs, rather than their reductions, was given by Weiss, Tabor, and Carnevale (WTC)[59]. The ARS conjecture and its variant by WTC are now taken to be almost self-evident because they have been formally verified for every known analytic soliton equation [45, 58, 51] (where analytic means that the equation is, or may be written to be, locally analytic in the dependent variable and its derivatives). Previously unknown integrable versions of the soliton equations [30, 15, 28] have been identified by the use of the conjecture. The conjecture has also been extrapolated to identify integrable ODEs [9, 57].

Rigorous results supporting the conjecture exist for ODEs with special symmetries or symplectic structure [60, 61]. Necessary conditions for possessing the Painlevé property [17, 29] have also been derived for general semilinear analytic second-order PDEs. No new integrable PDEs were found — suggesting that in this class at least, there is a one-to-one correspondence (modulo allowable transformations) between integrable equations and those possessing the Painlevé property. Moreover, proofs of weakened versions of the conjecture exist [5, 49]. These results point strongly to the truth of the ARS conjecture. Nevertheless, the conjecture has not yet been proved.

The main aim of this paper is to describe methods for investigating the singularity structure of solutions of ODEs and PDEs. These may be divided into two classes, those that parallel methods for analysing *regular* singular points and those that parallel techniques for *irregular* singular points of linear ODEs.

The first class of methods has been widely used formally. The most popular procedure is to expand every solution of the differential equation of interest in an infinite series near a movable singularity of the equation [46], i.e. the solution $u(z)$ is expanded as

$$u(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^{n+\rho} \tag{1.5}$$

where z_0 is the arbitrary location of a singularity and ρ is the leading power that needs to be found. Such an expansion is often called a Painlevé expansion. The equation is assumed to have the Painlevé property if the series is self-consistent, single-valued, and contains a sufficient number of degrees of freedom to describe all possible solutions or the general solution. These demands yield necessary conditions for the Painlevé property to hold. The series and the expansion techniques are analogues of the usual Fröbenius (or Fuchsian) expansion procedure for linear ODEs. This procedure was extended to PDEs by [59].

These techniques are, in general, not sufficient to prove that a differential equation has the Painlevé property. For example, even if the only possible formal solutions are Laurent series, the poles indicated by these series may accumulate elsewhere to give rise to a worse (branched) singularity.

Painlevé gave sufficient conditions to show that his eponymous equations have the Painlevé property. However, his proof is not widely understood. We describe briefly here an alternative, direct, method of proof due to Joshi and Kruskal [34]. To gain sufficiency, we showed that the solutions of the Painlevé equations possess convergent Laurent expansions around every movable singularity, and moreover, (in any given bounded region) the radius of convergence of each series is uniformly bounded below. In other words, the poles of any solution cannot coalesce to form a more complicated singularity elsewhere (in the finite plane).

For nonlinear PDEs, the question of how to get sufficient conditions for the Painlevé property is still open. Nevertheless, partial results are now known. These make the WTC analogue of the Fröbenius method rigorous and go some way toward proving that a given PDE has the Painlevé property. We describe the results due to Joshi and Petersen [35, 36] and Joshi and Srinivasan [38] in section 2. An alternative approach to the convergence of the Painlevé expansions for PDEs has also been developed by Kichenassamy and Littman [40, 41].

Another difficulty with the analysis of singularity structure is that the Painlevé expansions can miss some solutions. This may happen, for example, when the number of degrees of freedom in the series is less than the order of the differential equation. Perturbations of such series often reveal that the missing degrees of freedom lie in terms that occur (paradoxically) before the leading term. For reasons explained in Section 2, such terms are called *negative resonances*. In other cases, perturbations reveal no additional degrees of freedom at all. We call the latter series *defective*.

How can we deduce the singular behaviours of solutions that are missed by the Painlevé expansions? We provide an answer based on irregular-singular analysis for linear ODEs [8] and illustrate it through two important examples. The first example is the Chazy equation [1], a third-order ODE, whose general solutions have movable natural barriers. The Painlevé expansion of the solution of the Chazy equation contains only two arbitrary constants. The second example is a fourth-order ODE first studied by Bureau. This example has two families of Painlevé expansions, one of which has negative resonances and one that is defective. In section 3, we show how the Painlevé expansions can be extended through exponential (or WKB-type) perturbations.

Conte, Fordy, and Pickering [16] have followed an alternative approach. Their perturbations of Painlevé expansions involve Laurent series with no leading term (i.e. an infinite number of negative powers). As pointed out by one of us, this is well defined only in an annulus where the expansion variable is lower-bounded away from the singularity. Conte *et al.* demand that each term of such a perturbation must be single-valued. Therefore, their procedure requires a possibly infinite number of conditions to be checked for the Painlevé property. Our approach overcomes this problem.

For linear differential equations, in general, the analysis near an irregular singularity yields asymptotic results, i.e. asymptotic behaviours along with their domains of asymptotic validity near the singularity. The latter is crucial in this description. For example, it is well known that the Airy function $Ai(x)$ which solves the ODE

$$y'' = xy,$$

has the asymptotic behaviour

$$Ai(x) \approx \frac{1}{2\sqrt{\pi}x^{1/4}} \exp(-2x^{3/2}/3) \quad \text{as} \quad |x| \to \infty, \quad |\arg x| < \pi$$

near the irregular singular point at infinity. (See [8, 52].) Note that the asymptotic behaviour of $Ai(x)$ is apparently multivalued but the function itself is single-valued everywhere. (In fact, $Ai(x)$ is entire, i.e. it is analytic throughout the whole complex x-plane.)

The resolution of this apparent paradox lies in the angular width of the sector of validity of the above behaviour, which is strictly less than 2π. To describe $Ai(x)$ in the whole plane, we need its asymptotic behaviour in regions that include the line $|\arg(x)| = \pi$. (Such behaviours are well known. See e.g. [8].) These, together with the behaviour given above, show that the analytic continuation of $Ai(x)$ along a large closed curve around infinity is single-valued. Therefore, the global asymptotic description is not actually multivalued.

On the other hand, suppose an asymptotic behaviour is multivalued and its sector of validity extends further than 2π. Then there are (at least) two asymptotic descriptions of a solution at the same place (near an irregular singularity). This violates the uniqueness of asymptotic description of a solution, unless the solution is itself multivalued. Therefore, such an asymptotic behaviour is an indication that the solution cannot satisfy the requirements of the Painlevé property. The Bureau equation we study in Section 3 provides an example of this case.

Such results form an important extension of the usual tests for the Painlevé property. However, there is no denying that many fundamental questions remain open in this area, even at a formal level. For example, the Painlevé property is easily destroyed by straightforward transformations of the dependent variable(s). (E.g. A solution $u(z)$ of an ODE with movable simple poles is transformed to a function $w(z)$ with movable branch points under $u \mapsto w^2$.) An extension of the Painlevé property called the poly-Painlevé property has been proposed by Kruskal [45, 58] to overcome these difficulties. It allows solutions to be branched around movable singularities so long as a solution is not densely valued at a point. However, such developments lie outside the scope of this paper and we refer the reader to [45] for further details and references.

Other major problems remain. One is to extend the classification work of Painlevé and his colleagues to other classes of differential equations. Cosgrove has accomplished the most comprehensive extensions in recent times [17]. The universal method of classification called the α-method is based on asymptotic ideas (see Section 2). Asymptotic limits of differential equations can illuminate such studies.

The Painlevé equations are well known to have asymptotic limits to other equations with the Painlevé property. These limits are called *coalescence* limits because singularities of the equation merge under the limit. In Section 4, we describe the well known coalescence limits of the Painlevé equations and show that these limits also occur for integrable PDEs.

Throughout this article, solutions of differential equations assumed to be complex-valued functions of complex variables.

2 Nonlinear-Regular-Singular Analysis

In this section, we survey the main techniques used to study the Painlevé property. These range from the α-method to the widely used formal test known as the Painlevé test.

Consider the second-order linear ODE

$$u''(z) + p(z)u'(z) + q(z)u(z) = 0,$$

where primes denote differentiation with respect to z. Fuchs' theorem [8] states that u can only be singular (nonanalytic) at points where p and q are singular. Such singularities are called *fixed* because their positions are determined *à priori* (before solving the equation) and their locations remain unchanged throughout the space of all possible solutions.

However, fixed singularities are not the only possibilities for nonlinear equations. Consider the Riccati equation

$$u''(z) + u^2(z) = 0.$$

It has the general solution

$$u(z) = \frac{1}{z - z_0},$$

where z_0 is an arbitrary constant. If, for example, the initial condition is $u(0) = 1$, then $z_0 = -1$. If the initial condition is changed to $u(0) = 2$, then z_0 moves to $z_0 = -1/2$. In other words, the location of the singularity at z_0 *moves* with initial conditions. Such singularities are called *movable.*

Nonlinear equations exhibit a vast range of types of movable singularities. Some examples are given in the Table 1 (where k and z_0 are arbitrary constant parameters).

	Equation	General Solution	Singularity Type
Table 1: Examples of Possible Singular Behaviour			
1.	$y' + y^2 = 0$	$y = (z - z_0)^{-1}$	simple pole
2.	$2yy' = 1$	$y = \sqrt{z - z_0}$	branch point
3.	$y'' + y'^2 = 0$	$y = \ln(z - z_0) + k$	logarithmic branch point
4.	$yy'' + y'^2(y/y' - 1) = 0$	$y = k \exp\left([z - z_0]^{-1}\right)$	isolated essential singularity
5.	$(1 + y^2)y'' + (1 - 2y)y'^2 = 0$	$y = \tan\left(\ln(k[z - z_0])\right)$	nonisolated essential singularity
6.	$(y'' + y^3y')^2 = y^2y'^2(4y' + y^4)$	$y = k\tan[k^3(x - x_0)]$ or $y = \Big((4/3)/(x - x_0)\Big)^{1/3}$	pole / branch point

A normalized ODE, i.e. one that is solved for the highest derivative, such as

$$y^{(n)} = F\Big(y^{(n-1)}, \ldots, y', y, z\Big), \tag{2.6}$$

gives rise to possible singularities in its solutions wherever F becomes singular. (Where F is analytic, and regular initial data are given, standard theorems show that an analytic solution must exist.) Note that these singularities may include the points at infinity in y (or its derivatives) and z, which we denote by $y = \infty$ (or $y' = \infty$ etc), $z = \infty$. The singularities of

F, therefore, denote possible singularities in the solution(s). They may be divided into two classes: those given by values of z alone and those involving values of y or its derivatives. The former are determined *à priori* for all solutions. Therefore, they can only give rise to *fixed* singularities. To find movable singularities, we therefore need to investigate the singular values of F that involve y (or its derivatives). Similar statements can be made in the case of PDEs.

For example, the Riccati equation

$$y' = -y^2/z =: F(y, z).$$

has a right side F with two singularities given by $z = 0$ and $y = \infty$. The general solution is

$$y(z) = \frac{1}{\log(z/z_0)}.$$

It is clear that $z = 0$ is a fixed singularity (it stays the same for all initial conditions) whereas z_0 denotes a movable singularity where y becomes unbounded.

Singularities of nonlinear ODEs need not only occur at points where y is unbounded. Example 2 of Table 1 indicates possible movable singularities at points where $y = 0$. The solution shows that these are actually movable branch points.

These considerations show that singular values of the normalized differential equation lie at the base of the solutions' singularity structure. Techniques for investigating singularity structure usually focus on these singular values.

In the first three subsections below, we describe common definitions of the Painlevé property, and the two major techniques known as the α-method and the Painlevé test for deriving necessary conditions of the Painlevé property. In the subsequent three subsections below, we discuss the need for sufficiency conditions, the direct method of proving the Painlevé property, and convergence-type results for PDEs.

2.1 The Painlevé Property

The actual definition of the Painlevé property has been subject to some variation. There are three definitions in the literature.

Definition 2.1 *An ODE is said to possess*

1. *the* specialized Painlevé property *if all movable singularities of all solutions are poles.*

2. *the* Painlevé property *if all solutions are single-valued around all movable singularities.*
3. *the* generalized Painlevé property *if the general solution is single-valued around all movable singularities.*

(The qualifiers "specialized" and "generalized" are not usually used in the literature.) The first property defined above clearly implies the others. This property was also the first one investigated (by ARS) in recent times. It is the property possessed by the six Painlevé equations.

The second, more general, definition above is the one used by Painlevé in his work on the classification of ODEs. It allows, for example, movable unbranched essential singularities in any solution. Of the examples in Table 1, equations 1 and 4 have the Painlevé property; equation 1 also has the specialized Painlevé property. The remaining equations have neither property.

The third property is the most recently proposed variation, although there is evidence that Chazy assumed it in investigating ODEs of higher (≥ 1) degree or order (≥ 2). The sixth example given in Table 1 satisfies neither of the first two properties above because the special solution $(4/3/(x-x_0))^{1/3}$ has movable branch points around which the solution is multivalued. However, it does satisfy the generalized Painlevé property because the general solution $k\tan[k^3(x-x_0)]$ is meromorphic.

Most of the known techniques for investigating the Painlevé property have their origin in the classical work of Painlevé and his colleagues. They classified ODEs of the form

$$u'' = F(z; u, u'), \tag{2.7}$$

where F is rational in u and u' and analytic in z, according to whether or not they possess the Painlevé property.

They discovered that every equation possessing the Painlevé property could either be solved in terms of known functions (trigonometric functions, elliptic functions, solutions of linear ODEs, etc.) or transformed to one of the six equations now called the Painlevé equations (P_I–P_{VI}). They have standard forms that are listed below. (They are representatives of equivalence classes under Möbius transformations.) Their general solutions are higher transcendental functions.

The Painlevé Equations

$$\begin{aligned} u'' &= 6u^2 + z, \\ u'' &= 2u^3 + zu + \alpha, \end{aligned}$$

$$
\begin{aligned}
u'' &= \frac{1}{u}u'^2 - \frac{1}{z}u' + \frac{1}{z}(\alpha u^2+\beta) + \gamma u^3 + \frac{\delta}{u},\\
u'' &= \frac{1}{2u}u'^2 + \frac{3}{2}u^3 + 4zu^2 + 2(z^2-\alpha)u + \frac{\beta}{u},\\
u'' &= \left\{\frac{1}{2u} + \frac{1}{u-1}\right\}u'^2 - \frac{1}{z}u' + \frac{(u-1)^2}{z^2}\left(\alpha + \frac{\beta}{u}\right) + \frac{\gamma u}{z} + \frac{\delta u(u+1)}{u-1},\\
u'' &= \frac{1}{2}\left\{\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-z}\right\}u'^2 - \left\{\frac{1}{z} + \frac{1}{u-1} + \frac{1}{u-z}\right\}u'\\
&\quad + \frac{u(u-1)(u-z)}{z^2(z-1)^2}\left\{\alpha + \frac{\beta z}{u^2} + \frac{\gamma(z-1)}{(u-1)^2} + \frac{\delta z(z-1)}{(u-z)^2}\right\},
\end{aligned}
$$

Two main procedures were used in this work. The first is known as the α-method and the second is now called Painlevé analysis. Painlevé described the α-method in the following way.

> *Considérons un équation différentielle dont le coëfficient différentiel est une fonction (holomorphe pour $\alpha = 0$) d'un paramètre α. Si l'equation a ses points critiques fixes pour α quelconque (mais $\neq 0$), il en est de même, a fortiori pour $\alpha = 0$, et le développement de l'intégrale $y(x)$, suivant les puissances de α, a comme coëfficients des fonctions de x à points critiques fixes.*

(This extract is taken from footnote 3 on p.11 of [55]. In Painlevé's terminology, a critical point of a solution is a point around which it is multivalued.) In other words, suppose a parameter α can be introduced into an ODE in such a way that it is analytic for $\alpha = 0$. Then if the ODE has the Painlevé property for $\alpha \neq 0$, it must also have this property for $\alpha = 0$. We illustrate this method for the classification problem for first-order ODEs below.

The second procedure, called Painlevé analysis, is a method of examining the solution through formal expansions in neighbourhoods of singularities of the ODE. In particular, the procedure focusses on formal series expansions of the solution(s) in neighbourhoods of generic (arbitrary) points (not equal to fixed singularities). The series expansion is based on Frobenius analysis and usually takes the form given by Eqn(1.5).

As mentioned in the Introduction, this procedure was extended to PDEs by WTC (Weiss, Tabor, and Carnevale) [59]. For PDEs, the above definitions of the Painlevé property continue to hold under the interpretation that a *movable singularity* means *noncharacteristic analytic movable singularity manifold*.

A noncharacteristic manifold for a given PDE is a surface on which we can freely specify Cauchy data. The linear wave equation,

$$u_{tt} - u_{xx} = 0, \tag{2.8}$$

has the general solution

$$u(x,t) = f(t-x) + g(t+x),$$

where f and g are arbitrary. By a suitable choice of f and g we can construct a solution u with any type of singularity on the curves $t-x=k_1$, $t+x=k_2$, for arbitrary constants k_1, and k_2. These lines are characteristic manifolds for equation (2.8). This example illustrates why the Painlevé property says nothing about the singular behaviour of solutions on characteristic singularity manifolds.

The WTC procedure is to expand the solutions $u(x,t)$ of a PDE as

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)\Phi^{n+\rho}, \tag{2.9}$$

near a noncharacteristic analytic movable singularity manifold given by $\Phi = 0$. (This extends in the obvious way for functions of more than two variables.) The actual expansion can be simplified by using information specific to the PDE about its characteristic directions. For example, for the KdV equation, noncharacteristic means that $\Phi_x \neq 0$. Hence by using the implicit function theorem near the singularity manifold, we can write

$$\Phi(x,t) = x - \xi(t),$$

where $\xi(t)$ is an arbitrary function. This is explored further in subsection 2.3.2 below.

In some cases, the form of the series Eqn(1.5) (or ((2.9)) needs modification. A simple example of this is the ODE

$$u''' = 2(u')^3 + 1. \tag{2.10}$$

Here $v = u'$ is a Jacobian elliptic function with simple poles of residue ± 1. (See [7].) Hence a series expansion of $u(z)$ around such a singularity z_0, say, must start with $\pm\log(z-z_0)$. The remainder of the series is a power series expansion in powers of $z-z_0$. In such cases, the Painlevé property holds for the new variable v.

2.2 The α-Method

In this section, we illustrate the α-method by using it to find all ODEs of the form

$$u' = \frac{P(z,u)}{Q(z,u)} \tag{2.11}$$

possessing the Painlevé property, where P and Q are analytic in z and polynomial in u (with no common factors). The first step of the α-method is to introduce a small parameter α through a transformation of variables in such a way that the resulting ODE is analytic in α. However, the transformation must be suitably chosen so that the limit $\alpha \to 0$ allows us to focus on a movable singularity. This is crucial for deducing necessary conditions for the Painlevé property.

We accomplish this by using dominant balances of the ODE near such a singularity. (See [43, 8] for a definition and discussion of the method of dominant balances.)

If Q has a zero of multiplicity m at $u = a(z)$ then, after performing the transformation $u(z) \mapsto u(z) + a(z)$, equation (2.11) has the form

$$u^m u' = f(z, u), \tag{2.12}$$

where f is analytic in u at $u = 0$ and $f(z, 0) \not\equiv 0$ (since P and Q have no common factors). Choose z_0 so that $\kappa := f(z_0, 0) \neq 0$ and define the transformation

$$u(z) = \alpha U(Z), \quad z = z_0 + \alpha^n Z,$$

where n is yet to be determined and α is a small (but nonzero) parameter. Note that this is designed to focus on solutions that become close to the singular value $u = 0$ of the equation somewhere in the z-plane.

Equation (2.12) then becomes

$$\alpha^{m+1-n} U^m \frac{dU}{dZ} = f(z_0 + \alpha^n Z, \alpha U) = \kappa + O(\alpha).$$

This equation has a dominant balance when $n = m+1$. In this case the limit as $\alpha \to 0$ gives

$$U^m \frac{dU}{dZ} = \kappa$$

which has the exact solution

$$U(Z) = \{(m+1)\kappa Z + C\}^{\frac{1}{m+1}},$$

where C is a constant of integration. This solution has a movable branch point at $Z = -C/(\kappa(m+1))$ for all $m > 0$. Therefore, Eqn (2.11) cannot possess the Painlevé property unless $m = 0$, i.e. Q must be independent of u. That is, to possess the Painlevé property, Eqn (2.11) must necessarily be of the form

$$u' = a_0(z) + a_1(z)u + a_2(z)u^2 + \cdots + a_N(z)u^N, \tag{2.13}$$

for some nonnegative integer N.

The standard theorems of existence and uniqueness fail for this equation wherever u becomes unbounded. To investigate what happens in this case, we transform to $v := 1/u$. (Note that the Painlevé property is invariant under such a transformation.) Eqn (2.13) then becomes

$$v' + a_0(z)v^2 + a_1(z)v + a_2(z) + a_3(z)v^{-1} + \cdots + a_N(z)v^{2-N} = 0.$$

But this equation is of the form (2.11) and so it can only possess the Painlevé property if $N = 2$.

In summary, for Eqn (2.11) to possess the Painlevé property, it must necessarily be a Riccati equation:

$$u' = a_0(z) + a_1(z)u + a_2(z)u^2. \tag{2.14}$$

To show that this is also sufficient, consider the transformation

$$u = -\frac{1}{a_2(z)}\frac{w'}{w}$$

which linearizes Eqn (2.14)

$$a_2 w'' - (a_2' + a_1 a_2)w' + a_0 a_2^2 w = 0.$$

By Fuchs theorem [8], the singularities of any solution w can only occur at the singularities of $(a_2' + a_1 a_2)/a_2$ or $a_0 a_2$. These are fixed singularities. Hence the only movable singularities of u occur at the zeroes of w. Since w is analytic at its zeroes, it follows that u is meromorphic around such points. That is, Eqn (2.14) has the Painlevé property.

2.3 The Painlevé Test

Here we illustrate the widely used formal tests for the Painlevé property for ODEs and PDEs by using examples.

2.3.1 ODEs

Consider a class of ODEs given by

$$u'' = 6u^n + f(z), \tag{2.15}$$

where f is (locally) analytic and $n \geq 1$ is an integer (the cases $n = 0$ or 1 correspond to linear equations).

Standard theorems that yield analytic solutions fail for this equation wherever the right side becomes singular, i.e. where either $f(z)$ or u becomes unbounded. We concentrate on the second possibility to find movable singularities. This means that the hypothesized expansion Eqn(1.5) must start with a term that blows up at z_0. To find this term, substitute

$$u(z) \sim c_0(z - z_0)^p, \qquad z \to z_0,$$

where $\Re(p) < 0$, $c_0 \neq 0$, into Eqn (2.15). This gives the dominant equation

$$c_0 p(p-1)(z - z_0)^{p-2} + O\left((z - z_0)^{p-1}\right) = 6c_0^2(z - z_0)^{np} + O\left((z - z_0)^{np+1}\right). \tag{2.16}$$

The largest terms here must balance each other (otherwise there is no such solution). Since $c_0 \neq 0$ and $p \neq 0$ or 1, we get

$$p - 2 = np, \quad \Rightarrow \quad p = \frac{-2}{n-1}.$$

If p is not an integer, then u is branched at z_0. Hence, the only $n > 1$ for which equation (2.15) can possess the Painlevé property are $n = 2$ and $n = 3$.

We will only consider the case $n = 2$ here for conciseness. The case $n = 3$ is similar. (The reader is urged to retrace the following steps for the case $n = 3$.)

If $n = 2$ then $p = -2$ (which is consistent with our assumption that $\Re(p) < 0$). Then equation (2.16) becomes

$$6c_0(z - z_0)^{-4} = 6c_0^2 + O\left((z - z_0)^{-3}\right),$$

which gives $c_0 = 1$. Hence the hypothesized series expansion for u has the form

$$u(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^{n-2}. \tag{2.17}$$

The function $f(z)$ can also be expanded in a power series in $z - z_0$ because by assumption it is analytic. Doing so and substituting expansion (2.17) into eqn(2.15) gives

$$\begin{aligned}
&\sum_{n=0}^{\infty}(n-2)(n-3)c_n(z - z_0)^{n-4} \\
= \;& 6\sum_{i,j=0}^{\infty} c_i c_j (z - z_0)^{i+j-4} + \sum_{m=0}^{\infty} \frac{1}{m!} f^{(m)}(z_0)(z - z_0)^m \\
= \;& 6(z - z_0)^{-4} + 12c_1(z - z_0)^{-3} \\
& + 6(c_1^2 + 2c_2)(z - z_0)^{-2} + 12(c_3 + c_1 c_2)(z - z_0)^{-1} \\
& + \sum_{n=4}^{\infty}\left\{6\sum_{m=0}^{n} c_m c_{n-m} + \frac{1}{(n-4)!} f^{(n-4)}(z_0)\right\}(z - z_0)^{n-4}.
\end{aligned}$$

Equating coefficients of like powers of $(z - z_0)$ we get $c_1 = 0$, $c_2 = 0$, $c_3 = 0$, and

$$(n-2)(n-3)c_n = 6\sum_{m=0}^{n} c_m c_{n-m} + \frac{1}{(n-4)!} f^{(n-4)}(z_0), \quad (n \geq 4).$$

Note that c_n appears on both sides of this equation. Solving for c_n, we find

$$(n+1)(n-6)c_n = 6\sum_{m=1}^{n-1} c_m c_{n-m} + \frac{1}{(n-4)!} f^{(n-4)}(z_0). \tag{2.18}$$

For each $n \neq 6$, this relation defines c_n in terms of $\{c_m\}_{0\leq m<n}$. However, for $n = 6$, the coefficient of c_n vanishes and equation (2.18) fails to define c_6. If the right side also vanishes, c_6 is arbitrary. However, if the right side does not vanish there is a contradiction which implies that the series (2.17) cannot be a formal solution of eqn(2.15).

In that second case, the expansion can be modified to yield a formal solution by inserting appropriate logarithmic terms starting at the index $n = 6$. (This is also the case for Frobenius expansions when the indicial exponents differ by an integer – see [8]. See [41] also for a rigorous study of equations admitting such algebraico-logarithmic expansions in several variables.) In such a case, logarithmic terms appear infinitely often in the expansion and cannot be transformed away (as in Eqn(2.10) above). They therefore indicate multivaluedness around movable singularities.

That is, Eqn(2.15) fails the Painlevé test unless the right side of Eqn(2.18) vanishes at $n = 6$. This condition reduces to

$$f''(z_0) = 0.$$

However, since z_0 is arbitrary, this implies $f'' \equiv 0$. That is, $f(z) = az + b$ for some constants a, b. If $a = 0$, this equation can be solved in terms of (Weierstrass) elliptic functions. Otherwise, translating z and rescaling u and z gives

$$u'' = 6u^2 + z, \tag{2.19}$$

which is the first Painlevé equation.

The index of the free coefficient, c_6, in the above expansion is called a *resonance*. The expansion contains two arbitrary constants, c_6 and z_0, which indicates that it captures the generic singular behaviour of a solution (because the equation is second order).

There is a standard method for finding the location of resonances which avoids calculation of all previous coefficients. We illustrate this method here for P_I. After determining the leading order behaviour, substitute the perturbation

$$u \sim (z - z_0)^{-2} + \cdots + \beta(z - z_0)^{r-2}$$

where $r > 0$ into equation (2.19). Here β plays the role of the arbitrary coefficient. To find a resonance r, we collect terms in the equation that are linear in β and demand that the coefficient of β vanishes. This is equivalent to demanding that β be free. The resulting equation

$$(r + 1)(r - 6) = 0,$$

is called the resonance equation and is precisely the coefficient of c_r on the left side of equation (2.18).

The positive root $r = 6$ is precisely the resonance we found earlier. The negative root $r = -1$, often called the universal resonance, corresponds to the translation freedom in z_0. (Consider $z_0 \mapsto z_0 + \epsilon$. Taking $|\epsilon| < |z - z_0|$, and expanding in ϵ shows that $r = -1$ does correspond to an arbitrary perturbation.)

Note, however, that $r = -1$ is not always a resonance. For example, consider an expansion that starts with a nonzero constant term, such as

$$1 + a_1(z - z_0) + \ldots .$$

Perturbation of z_0 does not add a term corresponding to a simple pole to this expansion.

If any resonance is not an integer, then the equation fails the Painlevé test. The role played by other negative integer resonances is not fully understood. We explore this issue further through irregular singular point theory in Section 3.

For each resonance, the *resonance condition* needs to be verified, i.e. that the equation at that index is consistent. These give rise to necessary conditions for the Painlevé property. If all nonnegative resonance conditions are satisfied and all formal solutions around all generic arbitrary points z_0 are meromorphic, the equation is said to pass the Painlevé test.

This procedure needs to be carried out for evey possible singularity of the normalized equation. For example, the sixth Painlevé equation, P_{VI}, has four singular values in u, i.e. $u = 0$, 1, z and ∞. The expansion procedure outlined above needs to be carried out around arbitrary points where u approaches each such singular value. (Table 2 in Section 4 lists all singular values of the Painlevé equations.)

2.3.2 PDEs

In this subsection, we illustrate the WTC series expansion technique with an example. Consider the variable coefficient KdV equation,

$$u_t + f(t)uu_x + g(t)u_{xxx} = 0. \tag{2.20}$$

Let $\phi(x,t)$ be an arbitrary holomorphic function such that $S := \{(x,t) : \phi(x,t)\} = 0$ is noncharcteristic. The fact that S is noncharacteristic for equation (2.20) means that

$$\phi_x \neq 0 \tag{2.21}$$

on S. By the implicit function theorem, we have locally $\phi(x,t) = x - \xi(t)$ for some arbitrary function $\xi(t)$.

We begin by substituting an expansion of the form

$$u(x,t) = \sum_{n=0}^{\infty} u_n(t)\phi^{n+\alpha} \tag{2.22}$$

into equation (2.20). The leading order terms give $\alpha = -2$. Equating coefficients of powers of ϕ gives

$$\begin{aligned}
n = 0 &: \quad u_0 = -12g/f, && (2.23)\\
n = 1 &: \quad u_1 = 0, && (2.24)\\
n = 2 &: \quad u_2 = \xi'/f && (2.25)\\
n = 3 &: \quad u_3 = u_0'/(fu_0), && (2.26)\\
n \geq 4 &: \quad (n+1)(n-4)(n-6)gu_n && (2.27)\\
&= -f\sum_{k=0}^{n-4}(k+1)u_{n-k-3}u_{k+3} && (2.28)\\
&\quad +(n-4)\xi' u_{n-2} - u_{n-3}' && (2.29)
\end{aligned}$$

Arbitrary coefficients can enter at $n = 4$, 6 if the recursion relation is consistent. Consistency at $n = 6$ is equivalent to

$$\left(\frac{u_0'}{fu_0}\right)^2 + \frac{1}{f}\left(\frac{u_0'}{fu_0}\right)_t = 0.$$

This implies that

$$g(t) = f(t)\left\{a_0 \int^t f(s)ds + b_0\right\},$$

where a_0 and b_0 are arbitrary constants. In this case, Eqn(2.20) can be transformed exactly to the KdV equation (see Grimshaw [22]).

In particular, for the usual form of the KdV ($f(t) = 6$, $g(t) = 1$) we have the formal series expansion

$$\begin{aligned} u(x,t) &= \frac{-2}{\phi^2} + \frac{\xi'(t)}{6} + u_4(t)\phi^2 \\ &\quad -\xi''(t)\phi^3 + u_6(t)\phi^4 + O(\phi^5). \end{aligned} \tag{2.30}$$

Questions of convergence and well-posedness, i.e. continuity of the solution as the arbitrary functions ($\xi(t)$, $u_4(t)$, $u_6(t)$) vary, are discussed in Section 2.6 below.

2.4 Necessary versus sufficient conditions for the Painlevé property

The methods we described above can only yield necessary conditions for the Painlevé property. Here we illustrate this point with an example (due to Painlevé) which does not possess the Painlevé property, but for which the Painlevé test indicates only meromorphic solutions.

Consider the ODE

$$(1 + u^2)u'' + (1 - 2u)u'^2 = 0 \tag{2.31}$$

(see Ince [27]). The singularities of this equation are $u = \pm i$, $u = \infty$ and $u' = \infty$. Series expansions can be developed for solutions exhibiting each of the above singular behaviours and the equation passes the Painlevé test. This equation, however, has the general solution

$$u(z) = \tan\{\log[k(z - z_0)]\},$$

where k and z_0 are constants. For $k \neq 0$, u has poles at

$$z = z_0 + k^{-1}\exp\{-(n + 1/2)\pi\}$$

for every integer n. These poles accumulate at the movable point z_0, giving rise to a movable branched nonisolated essential singularity there. This example clearly illustrates the fact that passing the Painlevé test is not a guarantee that the equation actually possesses the Painlevé property.

This danger arises also for PDEs. The PDE

$$w_t = (1 + w^2)w_{xx} + (1 - 2w)w_x,$$

under the assumption

$$w(x,t) =: u(x)$$

reduces to the ODE above.

To be certain that a given differential equation possesses the Painlevé property, we must either solve it explicitly or implicitly (possibly through transformations to other equations known to have the property), or develop tests for sufficiency. Most results in the literature rely on the former approach. In the next section, we develop a method for testing sufficiency.

2.5 A Direct Proof of the Painlevé Property for ODEs

In this section we outline a direct proof given in (Joshi and Kruskal [31, 34]) that the Painlevé equations indeed possess the Painlevé property. The proof is based on the well known Picard iteration method (used to prove the standard theorems of existence and analyticity of solutions near regular points) modified to apply near singular points of the Painlevé equations. A recommended simple example for understanding the method of proof is

$$u'' = 6u^2 + 1$$

which is solved by Weierstrass elliptic functions.

Consider the initial value problem for each of the six Painlevé equations with bounded data for u and u' given at an ordinary point $z_1 \in \mathbb{C}$. (The point z_1 cannot equal 0 for $P_{III} - P_{VI}$, 1 for P_V or P_{VI}, or z_1 for P_{VI} — see Table 2.)

Table 2: Fixed and Movable Singularities of the Painlevé Equations

Equation	Fixed Singularities (z-value)	Movable Singularities (u-value)
P_I	$\{\infty\}$	$\{\infty\}$
P_{II}	$\{\infty\}$	$\{\infty\}$
P_{III}	$\{0, \infty\}$	$\{0, \infty\}$
P_{IV}	$\{\infty\}$	$\{0, \infty\}$
P_V	$\{0, \infty\}$	$\{0, 1, \infty\}$
P_{VI}	$\{0, 1, \infty\}$	$\{0, 1, z, \infty\}$

Standard theorems yield a (unique) solution U in any region in which the Lipschitz condition holds. However, they fail where the right side becomes unbounded, i.e. at its singular values. (See e.g. [14].) Since our purpose is to study the behaviour of the solution near its movable singularities, and these lie in the finite plane, we confine our attention to an arbitrarily large but bounded disk $|z| < B$ (where B is real and say > 1). For P_{III}, P_V, and

P_{VI} this must be punctured at the finite fixed singularities. Henceforth we concentrate on P_I for simplicity.

The ball $|z| < B$ contains two types of regions. Around each movable singularity, we select a neighbourhood where the largest terms in the equation are sufficiently dominant over the other terms. We refer to these as *special regions*. Outside these special regions, the terms remain bounded. Therefore, the ball resembles a piece of Swiss cheese, the holes (which may not be circular in general but in this case turn out to be nearly circular) being the special regions where movable singularities reside and the solid cheese being free of any singularity.

Starting at z_1 in (the cheese-like region of) the ball, we continue the solution U along a ray until we encounter a point z_2 on the edge of a special region. Inside the region, we convert P_I to an integral equation by operating successively on the equation as though only the dominant terms were present.

The dominant terms of

$$u'' = 6u^2 + z, \tag{2.32}$$

are u'' and $6u^2$. Integrating these dominant terms after multiplying by their integrating factor u', we get

$$\frac{u'^2}{2} = 2u^3 + zu - \int_{z_2}^{z} u\, dz + \bar{k}, \tag{2.33}$$

with

$$\bar{k} := \int_{z_1}^{z_2} u\, dz + k,$$

where the constant k (kept fixed below) is determined explicitly by the initial conditions.

Since u is large, u' does not vanish (according to Eqn(2.33)) and, therefore, there is a path of steepest ascent starting at z_2. We will use this idea to find a first point in the special region where u becomes infinite.

Let d be an upper bound on the length of the path of integration from z_2 to z and assume $A > 0$ is given such that

$$A^2 > 4B, A^2 > 4\pi B, A^2 > 4d, A^3 > 4|k|.$$

Assume that $|u| \geq A$ at z_2. Then Eqn(2.33) gives $u'(z_2) \neq= 0$. Taking the path of integration to be the path of steepest ascent, we can show that

$$|u'| > \sqrt{2}|u|^{3/2},$$

and that the distance to a point where $|u|$ becomes infinite is

$$d < \sqrt{2}A^{-1/2}.$$

(See page 193 of [34] for details.) So there is a first singularity encountered on this path which we will call z_0.

Now integrating the dominant terms once more (by dividing by $2u^3$, taking the square root and integrating from z_0) we get

$$u = \left(\int_{z_0}^{z} \left\{ 1 + \frac{1}{2u^3} \left(\bar{k} + zu - \int_{z_2}^{z} u \, dz \right) \right\}^{1/2} dz \right)^{-2}. \qquad (2.34)$$

Substituting a function of the form

$$u(x,t) = \frac{1}{(z - z_0)^2} + f(z)$$

where $f(z)$ is analytic at z_0 into the right side of equation (2.34) returns a function of the same form. Notice that, therefore, no logarithmic terms can arise.

It is worth noting that the iteration of the integral equation(2.34) gives the same expansion that we would have obtained by the Painlevé test. In particular, it generates the appropriate formal solution without any assumptions of its form and it point outs precisely where logarithmic terms may arise without additional investigations. (For example, try iteration of the integral equation with the term z on the right side of equation(2.32) replaced by z^2, i.e. with $zu - \int_{z_2}^{z} u \, dz$ replaced by $z^2 u - 2\int_{z_2}^{z} zu \, dz$ in Eqn(2.34).)

The remainder of the proof is a demonstration that the integral equation (2.34) has a unique solution meromorphic in a disk centred at z_0, that its radius is lower-bounded by a number that is independent of z_0, and that the solution is the same as the continued solution U. The uniformity of the lower bound is crucial for the proof. Uniformity excludes the possibility that the movable poles may accumulate to form movable essential singularities as in example(2.31).

Since the analytic continuation of U is accomplished along the union of segments of rays and circular arcs (skirting around the boundaries of successive special regions encountered on such rays) and these together with the special regions cover the whole ball $|z| < B$, we get a proof that the first Painlevé transcendent is meromorphic throughout the ball.

2.6 Rigorous Results for PDEs

Sufficient results for the Painlevé property of PDEs have been harder to achieve than ODEs. This is surprising because such results are lacking even for the most well known integrable PDEs. In this section, we describe some partial results towards this direction for the KdV equation.

Definition 2.2 *The* WTC data *for the KdV equation is the set* $\{\xi(t),\ u_4(t),\ u_6(t)\}$ *of arbitrary functions describing this Painlevé expansion.*

The following theorem proves that the series (2.30) converges for analytic WTC data.

Theorem 2.1 *(Joshi and Petersen [35, 36]) Given an analytic manifold* $S := \{(x,t) : x = \xi(t)\}$, *with* $\xi(0) = 0$, *and two arbitrary analytic functions*

$$\lim_{x\to\xi(t)} \Big(\frac{\partial}{\partial x}\Big)^4 [w(x,t)(x-\xi(t))^2], \quad \lim_{x\to\xi(t)} \Big(\frac{\partial}{\partial x}\Big)^6 [w(x,t)(x-\xi(t))^2]$$

there exists in a neighbourhood of $(0,0)$ *a meromorphic solution of the KdV equation (1.1) of the form*

$$w(x,t) = \frac{-12}{(x-\xi(t))^2} + h(x,t)$$

where h *is holomorphic.*

The next theorem provides us with a useful lower bound on the radius of convergence of this series.

Theorem 2.2 *(Joshi and Srinivasan [38]) Given WTC data* $\xi(t)$, $u_4(t)$, $u_6(t)$ *analytic in the ball* $B_{2\rho+\epsilon}(0) = \{t \in \mathbf{C} : |t| < 2\rho + \epsilon\}$, *let*

$$M = \sup_{|t|=2\rho} \{1, |\xi(t)|, |u_4(t)|, |u_6(t)|\}.$$

The radius of convergence $R_\rho = R$ *of the power series (1.5) satisfies*

$$R \geq \frac{\min\{1,\rho\}}{10M}.$$

This lower bound is used in [38] to prove the well-posedness of the WTC Cauchy problem. That is, the locally meromorphic function function described by the convergent series (2.30) has continuous dependence on the WTC data in the sup norm.

To date there is no proof that the Korteweg-de Vries equation possesses the Painlevé property. The main problem lies in a lack of methods for obtaining the global analytic description of a locally defined solution in the space of several complex variables. However, some partial results have been obtained.

The usual initial value problem for the KdV equation is given on the characteristic manifold $t = 0$. Well known symmetry reductions of the KdV equation (e.g. to a Painlevé equation) suggest that a generic solution must possess an infinite number of poles. WTC-type analysis shows that these can occur on noncharacteristic manifolds which intersect $t = 0$ transversely. These results suggest that only very special solutions can be entire (i.e. have no singularities) on $t = 0$.

Joshi and Petersen [37] showed that if the initial value

$$u(x,0) = u_0(x) := \sum_{n=0}^{\infty} a_n x^n,$$

is entire in x and, moreover, the coefficients a_n are real and nonnegative then there exists no solution holomorphic in any neighbourhood of the origin in $\mathbb{C}^2$ unless

$$u_0(x) = a_0 + a_1 x.$$

This result can be extended to the case of more general a_n under a condition on the growth of the function $u_0(x)$ as $x \to \infty$.

3 Nonlinear-Irregular-Singular Point Analysis

The Painlevé expansions cannot describe all possible singular behaviours of solutions of differential equations. In this section, we describe some extensions based on irregular-singular-point theory for linear equations.

The Painlevé expansions at their simplest are Laurent series with a leading term, and can, therefore, fail to describe solutions that possess movable isolated essential singularities. Consider the ODE

$$3u'u''' = 5(u'')^2 - (u')^2\frac{u''}{u} - \frac{(u')^4}{u^2},$$

which has the general solution

$$u(z) = \alpha \exp\left\{\beta(z-z_0)^{-1/2}\right\}.$$

Clearly u has a branched movable essential singularity. As suggested in [44], the Painlevé test can be extended to capture this behaviour by considering solutions that become exponentially large near z_0. To do this we expand

$$u(z) = a_{-1}(z)\mathrm{e}^{S}(z) + a_0(z) + a_1(z)\mathrm{e}^{-S(z)} + a_2(z)\mathrm{e}^{-2S(z)} + a_3(z)\mathrm{e}^{-3S(z)} + \dots,$$

where S and the a_n are generalized power series and S grows faster than any logarithm as z approaches z_0.

In other words, generalized expansions (those involving logarithms, powers, exponentials and their compositions) are necessary if we are to describe all possible singularities. These are asymptotic expansions which may fail to converge. They are in fact asymptotic expansions. We show in this section that they can nevertheless yield analytic information about solutions. We illustrate this with two main examples. The first is the Chazy equation and the second a fourth-order equation studied by Bureau.

3.1 The Chazy Equation

In this subsection, we examine the Chazy equation

$$y''' = 2yy'' - 3(y')^2. \tag{3.35}$$

This equation is exactly solvable through the transformation [12, 13]

$$z(t) := \frac{u_2(t)}{u_1(t)}, \quad y(z(t)) = \frac{6}{u_1(t)}\frac{du_1(t)}{dt} = 6\frac{d}{dt}\log u_1(t),$$

where u_1 and u_2 are two independent solutions of the Hypergeometric equation

$$t(t-1)\frac{d^2u}{dt^2} + \left(\frac{1}{2} - \frac{7}{6}t\right)\frac{du}{dt} - \frac{u}{144} = 0.$$

Following the work of Halphen [26], Chazy noted that the function $z(t)$ maps the upper half t-plane punctured at 0, 1, and ∞ to the interior of a circular triangle with angles $\pi/2$, $\pi/3$, and 0 in the z-plane (see, for example, Nehari [50]). The analytic continuation of the solutions u_1 and u_2 through one of the intervals $(0, 1)$, $(1, \infty)$, or $(-\infty, 0)$ corresponds to an inversion of the image triangle across one of its sides to a complementary triangle. Continuing this process indefinitely leads to a tessellation of either the interior or the exterior of a circle on the Riemann sphere This circle is a *natural barrier* in the sense that the solution can be analytically extended up to but not through it.

We will see below that any solution of equation (3.35) is single-valued everywhere it is defined. The general solution, however, possesses a movable natural barrier, i.e. a closed curve on the Riemann sphere whose location depends on initial conditions and through which the solution cannot be analytically continued.

Leading order analysis of equation (3.35) shows that near a pole,

$$y \sim -6(z-z_0)^{-1}, \quad \text{or} \quad y \sim A(z-z_0)^{-2},$$

where A is an arbitrary (but nonzero) constant. On calculating successive terms in this generalized series expansion we find only the exact solution

$$y(z) = \frac{A}{(z-z_0)^2} - \frac{6}{z-z_0}. \tag{3.36}$$

This solution has only two arbitrary constants, A and z_0, and clearly cannot describe all possible solutions of equation (3.35) which is third order. That is, solutions of the form (3.36) fail to capture the generic behaviour of the full space of solutions.

The absent degree of freedom may lie in a perturbation of this solution. Applying the usual procedure for locating resonances i.e. substituting the expression

$$y(z) \sim \frac{-6}{(z-z_0)^2} + \cdots + \beta(z-z_0)^{r-2}$$

into equation (3.35) and demanding that β be free, we find

$$(r+1)(r+2)(r+3) = 0.$$

I.e. we must have $r=-1$, $r=-2$ or $r=-3$. The case $r=-1$ corresponds to the fact that A is arbitrary in (3.36). The case $r=-2$ corresponds to the freedom in z_0. The case $r=-3$, however, indicates something more.

Since the usual Frobenius-type series fails to describe the general solution near a singular point, we turn to be a nonlinear analogue of irregular singular point theory. Arguing from analogy with the linear theory (see, for example, Bender and Orszag [8]) we seek a solution of the form

$$y(z) = \frac{A}{(z-z_0)^2} - \frac{6}{(z-z_0)} + \exp S(z), \tag{3.37}$$

where $\exp S(z)$ is regarded as small in a region near z_0 (generically, z_0 will be on the boundary of this region).

Since the Chazy equation is autonomous we can, without loss of generality, take $z_0 = 0$. For simplicity we also take $A = 1/2$. Substituting the expansion (3.37) into equation (3.35) gives

$$\begin{aligned} &S''' + 3S'S'' + S'^3 \\ &\quad = \left(\frac{1}{z^2} - \frac{12}{z}\right)\left(S'' + S'^2\right) + 6\left(\frac{1}{z^3} - \frac{6}{z^2}\right)S' \\ &\quad +6\left(\frac{1}{z^4} - \frac{4}{z^3}\right) + 2\left(S'' + S'^2\right)\mathrm{e}^S - 3S'^2\mathrm{e}^S. \end{aligned} \tag{3.38}$$

To ensure that $\exp(S)$ is exponential rather than algebraic, we must assume that $S'' \ll S'^2$, $S''' \ll S'^3$. Using these assumptions along with $\exp S \ll 1$,

$S' \gg 1$, and $z \ll 1$, equation (3.38) gives

$$S' \sim \frac{1}{z^2} - \frac{2}{z}.$$

Integration yields

$$y(z) \sim \frac{1}{2z^2} - \frac{6}{z} + \frac{k}{z^2}\mathrm{e}^{-1/z},$$

where k is an arbitrary constant; k represents the third degree of freedom that was missing from the Laurent series(3.36). Extending to higher orders in $\exp S(z)$, we obtain the double series

$$\begin{aligned} y(z) &= \frac{1}{2z^2} - \frac{6}{z} \\ &\quad + \frac{k}{z^2}\mathrm{e}^{-1/z}\left(1 + O(z)\right) + \frac{k^2}{8z^2}\mathrm{e}^{-2/z}\left(1 + O(z)\right) \\ &\quad + O\left(\frac{\mathrm{e}^{-3/z}}{z^2}\right). \end{aligned} \tag{3.39}$$

It can be shown that this series is convergent in a half-plane, given here by $\Re(1/z) > 0$.

This asymptotic series is valid wherever

$$|k \exp(-1/z)| \ll 1. \tag{3.40}$$

Suppose k is small. Put

$$z = -\xi + \eta,$$

where $\xi > 0$ to see whether the half-plane of validity can be extended. Then the condition (3.40) becomes

$$\frac{\xi}{\xi^2 + \eta^2} \ll \log\left(\frac{1}{|k|}\right).$$

By completing squares (after multiplying out the denominator) this can be rewritten as

$$(\xi - \delta)^2 + \eta^2 \gg \delta^2,$$

where

$$\delta := -\frac{1}{2\log|k|} > 0.$$

So asymptotically the region of validity of the series (3.39) lies outside a circle of radius δ centered at $-\delta$. This is the circular barrier present in the general solution of the Chazy equation. In summary, the exponential (or WKB-type) approach has led to a three parameter solution. Morover, this description is valid in a region bounded by a circular curve where it diverges.

3.2 The Bureau Equation

Bureau partially extended the classification work of Painlevé and colleagues to fourth-order equations. However, there were cases whose Painlevé property could not be determined within the class of techniques developed by Painlevé's school. One of these was

$$u'''' = 3u''u - 4u'^2, \tag{3.41}$$

which we will call the Bureau equation. In this subsection, we show that the general solution is multivalued around movable singularities by using exponential or WKB-type expansions based on irregular-singular-point theory.

It has been pointed out by several authors that Eqn(3.41) possesses two families of Painlevé expansions

$$u \approx \frac{a_\nu}{z^\nu} \tag{3.42}$$

(where we have shifted $z - z_0$ to z by using the equation's translation invariance) distinguished by

$$\begin{aligned} \nu &= 2, \quad a_2 = 60 \\ \nu &= 3, \quad a_3 \text{ arbitrary} \end{aligned}$$

with resonances given by

$$\begin{aligned} u &\approx \frac{a_\nu}{z^\nu} + \ldots + kz^{r-\nu} \\ \nu = 2 &\Rightarrow r = -3, -2, -1, 20 \\ \nu = 3 &\Rightarrow r = -1, 0. \end{aligned}$$

The case $\nu = 2$ has a full set of resonances (even though two are negative resonances other than the universal one). However, the case $\nu = 3$ is defective because its perturbation (in the class of Painlevé expansions) yields no additional degrees of freedom to the two already present in a_3 and z_0. It is, in fact, given by the two-term expansion

$$u = \frac{a}{z^3} + \frac{60}{z^2}.$$

We concentrate on this defective expansion in the remainder of this subsection. Since this expansion allows no perturbation in the class of conventional Painlevé expansions (which are based on regular-singular-point theory), we turn to perturbations of the form based on irregular-singular-point theory. Consider

$$u = \frac{a}{z^3} + \frac{60}{z^2} + \hat{u},$$

where

$$\hat{u} = \exp(S(z)), \;\; S' \gg \frac{1}{z}, z \ll 1.$$

(The assumption on S' is to assure that $\exp(S)$ is exponential rather than algebraic.) Substituting this into Eqn(3.41), we get

$$\begin{aligned} S'^4 \;\; &= \;\; \frac{3a}{z^3}S'^2 + \left(\frac{3a}{z^3}S'' + \frac{24a}{z^4}S' - 6S'^2S''\right) \\ &\quad + \left(\frac{36a}{z^5} + \frac{180}{z^2}S'^2 - 4S'S''' - 3S''^2\right) \\ &\quad + \left(\frac{960}{z^3}S' + \frac{180}{z^2}S'' - S''''\right) + \frac{1080}{z^4} \\ &\quad + (3S'' - S'^2)e^S \end{aligned} \tag{3.43}$$

The condition that $\exp(S)$ not be algebraic also implies that $S'^2 \gg S''$, $S'^3 \gg S'''$, and $S'^4 \gg S^{(IV)}$. So dividing Eqn(3.43) by S'^2, taking the square root of the equation, and expanding we get

$$S' = \frac{(3a)^{1/2}}{z^{3/2}} + \frac{31}{4z} + O(z^{-1/2}) \tag{3.44}$$

where we have used recursive substitution of the leading values of S' and S'' to get the term of order $1/z$. That is, we get

$$S = -2\frac{(3a)^{1/2}}{z^{1/2}} + \frac{31}{4}\log z + \text{const} + o(1).$$

Take one such solution, with say $a = 1/3$. Then the perturbed solution has expansion

$$u = \frac{1}{3z^3} + \frac{60}{z^2} + k_{\pm}z^{31/4}\exp\Big(-2/z^{1/2}\Big)\Big(1 + o(1)\Big), \tag{3.45}$$

where $k_{\pm}$ is an arbitrary constant. Note that there are two exponentials here (due to the two branches of the square root of z) and, therefore, $k_{\pm}$ represents two degrees of freedom. In the following, we consider only one of these solutions by fixing a branch of the square root in the exponential, say to be the one that is positive real on the positive real semi-axis in the z-plane. For short, we write $k_+ = k$.

Now consider the domain (or sector) of validity of this solution. Note that the neglected terms in its expansion contain a series of powers of $\exp(S)$

due to the nonlinear terms in Eqn(3.43). Therefore, for the expansion to be asymptotically valid, this exponential term must be bounded i.e.

$$\left| k z^{31/4} \exp\Big(-2/z^{1/2}\Big) \right| < 1$$
$$\Rightarrow |k| \exp\bigg(\Re\Big(-2/z^{1/2} + (31/4)\ln(z)\Big)\bigg) < 1 \tag{3.46}$$

We show below that the domain of validity given by this inequality contains a punctured disk (on a Riemann surface) whose angular width is larger than 2π.

Assume there is a branch cut along the negative semi-axis in the z-plane with $\arg(z) \in (-\pi, \pi]$. Consider $z^{1/2}$ in polar coordinates, i.e. $z^{1/2} = re^{i\theta}$, where $-\pi/2 < \theta < \pi/2$. The positive branch will then give real part

$$\Re\Big(-2/z^{1/2} + (31/4)\ln(z)\Big) = -2\,\frac{1}{r}\cos(\theta) + \frac{31\ln r}{4}.$$

Let $K := \ln|k|/2$. To satisfy Eqn(3.46), we must have

$$-\frac{2}{r}\cos(\theta) + \frac{31\ln r}{4} + 2K + o(1) < 0$$

for $r \ll 1$, i.e.

$$-\cos(\theta) < -\frac{31r\ln r}{8} - Kr\Big(1 + o(1)\Big). \tag{3.47}$$

Since r is small (note $\ln r < 0$), this can only be violated near $\theta = \pm\pi/2$. Fix r small. Expand $\theta = \pi/2 + \epsilon$. Then Eqn(3.47) gives

$$\epsilon < -\frac{31r\ln r}{8} - Kr\Big(1 + o(1)\Big) + O(\epsilon^3). \tag{3.48}$$

In particular, ϵ can be negative (so long as $|\epsilon| < 1$). A similar calculation can be made near $-\pi/2$.

These results show that the asymptotic validity of the solution given by Eqn(3.45) can be extended to a domain which is a disk of angular width $> 2\pi$. The small angular overlap is given by a sector of angular width 2ϵ where ϵ is bounded by $O(r\ln r)$ according to Eqn(3.48).

Let z_s be a point in this overlapping wedge with small modulus. At such a point, we have two asymptotic representations of u, one given by a prior choice of branch of $z_s^{1/2}$ and the other given by analytic continuation across the branch cut. If the true solution is single-valued in this domain, the choice of two asymptotic representations violates uniqueness. Therefore, the true solution must itself be multivalued. In other words, the exponential expansion shows that Bureau's equation cannot have the Painlevé property.

4 Coalescence Limits

In this section we examine asymptotic limits of integrable equations that preserve the Painlevé property. In the case of ODEs, such limits form the basis of Painlevé's α-test. They are useful in the identification of nonintegrable equations and may be useful for indentifying new integrable equations as limits of others.

Painlevé [56] noted that under the transformation

$$\begin{aligned} z &= \epsilon^2 x - 6\epsilon^{-10}, \\ u &= \epsilon y + \epsilon^{-5}, \\ \alpha &= 4\epsilon^{-15}, \end{aligned}$$

P_{II} becomes

$$y''(x) = 6y^2 + x + \epsilon^6 \left\{ 2y^3 + xy \right\}. \tag{4.49}$$

In the limit as ϵ vanishes, equation (4.49) becomes P_I. We write the above limiting process as $P_{II} \longrightarrow P_I$. Painlevé gave a series of such asymptotic limits which are summarized in Figure 1.

$$\begin{array}{ccccccccc} & & & & P_{III} & & & & \\ & & & \nearrow & & \searrow & & & \\ P_{VI} & \longrightarrow & P_V & & & & P_{II} & \longrightarrow & P_I \\ & & & \searrow & & \nearrow & & & \\ & & & & P_{IV} & & & & \end{array}$$

Figure 1: Asymptotic limits among the Painlevé equations.

Each of these asymptotic limits coalesce the singular u-values of the Painlevé equation (see Table 2) i.e. they coalesce movable singularities. In [24], Halburd and Joshi proved that in the $P_{II} \longrightarrow P_I$ limit, simple poles of opposite residue coalesce to form the double poles in solutions of P_I. They also proved that all solutions of P_I can be obtained as limits of solutions of equation (4.49):

Theorem 4.1 *Choose $x_0, \alpha, \beta \in \mathbf{C}$. Let y_I and y be maximally extended solutions of P_I and equation (4.49) respectively, both satisfying the initial value problem given by*

$$y(x_0) = \alpha, \qquad y'(x_0) = \beta.$$

Let $\Omega \subset \mathbf{C}$ be the domain of analyticity of y_I. Given any compact $K \subset \Omega$, $\exists r_K > 0$ such that y is analytic in (x, ϵ) for $x \in K$ and $|\epsilon| < r_K$. Moreover, $y \to y_I$ in the sup norm as $\epsilon \to 0$.

The series of asymptotic (or *coalescence*) limits given in Figure 1 by no means represents a complete list of such limits. The singular u-values of P_{IV} are 0 and ∞, corresponding to zeros and poles of the solutions respectively (the solutions of P_{IV} are meromorphic [55, 34]). The standard coalescence limit in which P_{IV} becomes P_{II} merges poles and zeros. However, the general solution of P_{IV} contains simple poles of oppositely signed residue which may be able to merge pairwise. An asymptotic limit coalescing these simple poles to form double poles does exist [32].

To see this, consider a transformation in which regions near infinity (where the poles are close to each other) are mapped to the finite plane in the limit $\epsilon \to 0$. It is necessary to rescale u to counter a cancellation of the oppositely signed poles. This leads to new variables x and $w(x)$ given by

$$\begin{aligned} u(z) &= \epsilon^p w(x), \\ z &= N + \epsilon^q x, \end{aligned}$$

where $p, q > 0$, $\epsilon \ll 1$ and $N \gg 1$ is to be found in terms of ϵ. Then P_{IV} becomes

$$\begin{aligned} w_{xx} &= \frac{w_x^2}{2w} + \frac{3}{2}\epsilon^{2(p+q)}w^3 + 4N\epsilon^{p+2q}w^2 + 8\epsilon^{p+3q}xw^2 + 2(N^2-\alpha)\epsilon^{2q}w \\ &\qquad +4N\epsilon^{3q}xw + 2\epsilon^{4q}x^2w + \frac{\beta\epsilon^{2(q-p)}}{w}. \end{aligned} \tag{4.50}$$

The only maximal dominant balance (*i.e.* a limiting state of the equation in which a maximal number of largest terms remains [8]) occurs when

$$q = p, \quad \text{and} \quad \alpha =: N^2 + a\epsilon^{-2q},$$

where a is a constant. Then setting the largest terms $N\epsilon^{p+2q}$ and $N\epsilon^{3q}$ to unity gives $N = \epsilon^{-3p}$. Without loss of generality redefine $\epsilon^p \mapsto \epsilon$. Then we get

$$N = \epsilon^{-3} \quad \text{and} \quad \alpha = \epsilon^{-6} + a\epsilon^{-2}.$$

Equation (4.50) then becomes

$$w_{xx} = \frac{w_x^2}{2w} 4w^2 + (2x-a)w + \frac{\beta}{w} + \frac{3}{2}\epsilon^4 w^3 + 8\epsilon^4 xw^2 + 2\epsilon^4 x^2 w,$$

or, in the limit $\epsilon \to 0$

$$w_{xx} = \frac{w_x^2}{2w} 4w^2 + (2x - a)w + \frac{\beta}{w},$$

which is Equation (XXXIV) (see p.340 of Ince [27]) in the Painlevé-Gambier classification of second-order differential equations (after a simple scaling and transformation of variables).

Coalescence limits exist among PDEs also. For example, it is straightforward to derive the transformation [25]

$$\begin{aligned} \tau &= t; \\ \xi &= x + \frac{3}{2\epsilon^2} t; \\ u(x,t) &= \epsilon U(\xi,\tau) - \frac{1}{2\epsilon}, \end{aligned}$$

which maps the modified Korteweg-de Vries equation (mKdV)

$$u_t - 6u^2 u_x + u_{xxx} = 0,$$

to

$$U_\tau - 6\epsilon^2 U^2 U_\xi + 6UU_\xi + U_{\xi\xi\xi} = 0,$$

which becomes the usual KdV equation in the limit $\epsilon \to 0$. An alternative method for obtaining the above asymptotic limit is to use the $P_{II} \to P_I$ limit. The mKdV equation is invariant under the scaling symmetry

$$u \mapsto \lambda^{-1} u, \quad t \mapsto \lambda^3 t, \quad x \mapsto \lambda x.$$

Define the canonical variables

$$z = \frac{x}{(3t)^{1/3}}, \quad w = \frac{1}{3}\log t, \quad u(x,t) = (3t)^{-1/3} y(z,w).$$

In terms of these variables, the mKdV equation becomes

$$(\underbrace{y_{zz} - 2y^3 - zy - \alpha}_{P_{II}})_z + y_w = 0,$$

where we have included the constant α to emphasize its relation to P_{II}. Now apply the asymptotic transformation used in the $P_{II} \to P_I$ limit, to determine how y and z transform, and transform w in such a way that it remains in the limiting form of the equation as $\epsilon \to 0$. In this way we arrive at an equation equivalent to the KdV equation, which has a reduction to P_I.

In [25] it is shown that the system

$$\begin{aligned} E_x &= \rho, \\ \widetilde{E}_x &= \tilde{\rho}, \\ 2N_t &= -(\rho\widetilde{E} + \tilde{\rho}E), \\ \rho_t &= NE, \\ \tilde{\rho}_t &= N\widetilde{E}, \end{aligned} \tag{4.51}$$

admits a reduction to the full P_{III} (P_{III} with all four constants $\delta \neq 0$, α, β, γ arbitrary). We note that if $\widetilde{E} = E^*$ and $\tilde{\rho} = \rho^*$, where a star denotes complex conjugation, we recover the unpumped Maxwell-Bloch equations;

$$E_x = \rho, \quad 2N_t + \rho E^* + \rho^* E = 0, \quad \rho_t = NE.$$

Consider solutions of system (4.51) of the form

$$\begin{aligned} E &= t^{-1}\varepsilon(z)w, \\ \widetilde{E} &= t^{-1}\tilde{\varepsilon}(z)w^{-1}, \\ N &= n(z), \\ \rho &= r(z)w, \\ \tilde{\rho} &= \tilde{r}(z)w^{-1}, \end{aligned}$$

where $z := \sqrt{xt}$ and $w := (x/t)^k$, k constant. Then

$$y(z) := \frac{\varepsilon(z)}{zr(z)}$$

solves P_{III} with constants given by

$$\alpha = 2(r\tilde{\varepsilon} - \tilde{r}\varepsilon + 4k), \quad \beta = 4(1+2k), \quad \gamma = 4(n^2 + r\tilde{r}), \quad \delta = -4.$$

Note that by rescaling y we can make δ any nonzero number.

Using the procedure outlined for mKdV → KdV, it can be shown [25] that the $P_{III} \to P_{II}$ limit induces an asymptotic limit in which the generalized unpumped Maxwell-Bloch system (4.51) becomes the dispersive water-wave equation (DWW)

$$u_{xxxx} + 2u_t u_{xx} + 4u_x u_{xt} + 6u_x^2 u_{xx} + u_{tt} = 0,$$

which is known to admit a reduction to P_{II} (Ludlow and Clarkson [47]). The $P_{II} \to P_I$ limit then gives DWW → KdV. Ludlow and Clarkson [47] have shown that DWW also admits a symmetry reduction to the full P_{IV}. The

$P_{IV} \to P_{II}$ limit then induces an asymptotic limit that maps DWW back to itself in a nontrivial way. The limit $P_{IV} \to P_{34}$, outlined above, induces another limit in which DWW is mapped to the KdV. All six Painlevé equations are known to arise as reductions of the self-dual Yang-Mills (SDYM) equations (Mason and Woddhouse [48]). Asymptotic limits between the Painlevé equations can be used to induce similar limits within the SDYM system (Halburd [23]).

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