Moments and Amplification II

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Set-up

$\mathscr{F}$ orthonormal basis of $S_k(\Gamma_0(q))$,

$$\Delta(m,n) = \sum_{f \in \mathscr{F}} \psi_f(m)\overline{\psi_f}(n).$$

We want to estimate the sum

$$\mathscr{B}(r,s) = \sum_m \sum_n \tau(m)\tau(n)\Delta(rm,sn)F(m,n),$$

where $\Delta(m,n) = \sum_{f\in\mathscr{F}} \psi_f(m)\overline{\psi_f}(n)$ and F smooth test function supported in $[M,2M]\times[N,2N]$ with $F^{(i,j)} \ll M^{-i}N^{-j}$.

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Theorem 2

Assume $(q,rs) = 1$ and $M,N \ll q^{1+\epsilon}$. Then

$$\mathscr{B}(r,s) \ll q^{\epsilon}[q(r,s)(rs)^{-\frac{1}{2}} + q^{\frac{11}{12}}(rs)^{\frac{3}{4}}](MN)^{\frac{1}{2}}$$

Petersson formula

For now $\mathscr{B}(s) \doteq \mathscr{B}(1, s)$. Using Petersson formula we got

$$\mathscr{B}(s) = (k-1)qT(0) + 2\pi i^k (k-1)q \sum_{c \equiv 0 \mod q} c^{-2} T(c),$$

with

$$T(0) = \sum_n \tau(sn)\tau(n)F(sn, n) \ll \left(\frac{MN}{s}\right)^{\frac{1}{2}} q^{\epsilon},$$

$$T(c) = c \sum_m \sum_n \tau(m)\tau(n)S(m, sn; c)J_{k-1}\left(\frac{4\pi\sqrt{smn}}{c}\right)F(m, n).$$

Transforming $T(c)$ into Ramanujan sums

By Jutila Poisson summation $T(c) = T^*(c) + T^-(c) + T^+(c)$ where

$$T^*(c) = \sum_n \tau(n) S(0, sn; c) G^*(n) \ll (c, s) MNc^\epsilon,$$

$$T^\pm(c) = \sum_m \sum_n \tau(m)\tau(n) S(0, sn \pm m; c) G^\pm(m, n)$$

where

$$G^-(x, y) = -2\pi \int Y_0\left(\frac{4\pi\sqrt{xy}}{c}\right) J_{k-1}\left(\frac{4\pi\sqrt{sxy}}{c}\right) F(x, y)$$

and G^+ is similar with $4K_0$ replacing $-2\pi Y_0$.

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Split

$$T^\pm(c) = \sum_h S(0, h, c)\, T_h^\pm(c)$$

where $T_h^\pm(c) = \sum_{sn \pm m = h} \tau(n)\tau(m) G^\pm(m, n)$.

A theorem from DFI: A quadratic divisor problem

Assume f is a smooth function on $\mathbb{R}^+ \times \mathbb{R}^+$ satisfying

$$x^i y^j f^{(i,j)}(x,y) \ll \left(1+\frac{x}{X}\right)^{-1}\left(1+\frac{y}{Y}\right)^{-1} P^{i+j}. \tag{1}$$

Define $\lambda_{aw} = 2\gamma + \log \frac{aw^2}{(a,w)^2}$, and

$$\Lambda_{abh}(x,y) = \frac{1}{ab}\sum_{w=1}^{\infty}\frac{(ab,w)}{w^2}S(h,0;w)(\log x - \lambda_{aw})(\log y - \lambda_{bw}).$$

Theorem 1

Suppose $h \neq 0$, $a, b \geq 1$ and $(a,b) = 1$. Then

$$\sum_{am \pm bn = h} \tau(m)\tau(n) f(am, bn) = \int_0^{\infty} f(x, \pm x \pm h)\Lambda_{abh}(x, \pm x \pm h)dx$$
$$+ O(P^{\frac{5}{4}}(X+Y)^{\frac{1}{4}}(XY)^{\frac{1}{4}+\epsilon}).$$

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Evaluation of $T_h^{\pm}(c)$

$sn + m = 0$ has no solution in positive integers so $T_0^{+}(c) = 0$. For now leave $T_0^{-}(c)$ as it is. For $h \neq 0$, use Theorem 1 with the test function $f^{\pm}(x,y) = G^{\pm}\left(x, \frac{y}{s}\right)$ satisfying (1) with $Y = sN$, $P = 1 + \sqrt{sMN}/c$, $X = c^2P^2M^{-1} > Y$.

Truncating the series defining $\Lambda_h(x,y)$ to $w < q$, we get

$$T_h^{-}(c) = \sum_{1 \leq w < q} \frac{(s,w)}{w^2} S(0,h;w)Y(h) + O((1+|h|/X)^{-2}P^{\frac{1}{4}}(sN)^{\frac{3}{4}}Mc^{\epsilon})$$

where

$$Y(h) = -2\pi \int\int [\log(h+sy) - \lambda_w][\log y - \lambda_{sw}] Y_0\left(\frac{4\pi}{c}\sqrt{(h+sy)x}\right) J_{k-1}\left(\frac{4\pi}{c}\sqrt{sxy}\right) F(x,y)dxdy.$$

and similarly for T_h^{+} with $-2\pi Y_0$ replaced with $4K_0$.

A Poisson-type lemma

Lemma 10.1

Let f be a $\mathcal{C}^2$ function on $\mathbb{R}$ such that $(1+x^2)f^{(\ell)}(x) \ll 1$. Then

$$\sum_h S(0,h;c)S(0,h;w)f(h) = \varphi((c,w)) \sum_u{}' \hat{f}\left(\frac{u(c,w)}{cw}\right).$$

Here $\sum{}'$ means the summation is restricted to $\left(u, \frac{cw}{(c,w)^2}\right) = 1$.

Idea: Split the summation in progressions $h \equiv a \mod [c,w]$ and apply Poisson summation to get sums involving

$$\sum_{a \mod [c,w]} S(0,a;c)S(0,a;w)e\left(-\frac{au}{[c,w]}\right),$$

which counts the number of solutions to a certain congruence condition.

Evaluation of $T^{\pm}(c)$

By the evaluation of T_h^-,

$$T^-(c) = \varphi(c)T_0^-(c) + \sum_{1\leq w<q}\sum_{h\neq 0} S(0,h;w)S(0,h;c)Y(h) + O\left(P^{\frac{9}{4}}(sN)^{\frac{3}{4}}c^{2+\epsilon}\right).$$

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Add and subtract the contribution from Y(0) then apply Lemma 10.1 for $Y(h)$ to get

$$T^-(c) = \varphi(c)T_0^-(c) - \varphi(c)\sum_{1\leq w<q}\varphi(w)\frac{(s,w)}{w^2}Y(0) + \sum_{1\leq w<q}\varphi((c,w))\frac{(s,w)}{w^2}{\sum_u}'\hat{Y}\left(\frac{u(c,w)}{cw}\right) + O\left(P^{\frac{9}{4}}(sN)^{\frac{3}{4}}c^{2+\epsilon}\right). \quad (2)$$

Evaluation of $T^{\pm}(c)$

By the evaluation of T_h^-,

$$T^-(c) = \varphi(c) T_0^-(c) + \sum_{1 \le w < q} \sum_{h \ne 0} S(0,h;w) S(0,h;c) Y(h) + O\left(P^{\frac{9}{4}} (sN)^{\frac{3}{4}} c^{2+\epsilon}\right).$$

Add and subtract the contribution from Y(0) then apply Lemma 10.1 for $Y(h)$ to get

$$T^-(c) = \varphi(c) T_0^-(c) - \varphi(c) \sum_{1 \le w < q} \varphi(w) \frac{(s,w)}{w^2} Y(0) + \sum_{1 \le w < q} \varphi((c,w)) \frac{(s,w)}{w^2} \sum_u{}' \hat{Y}\left(\frac{u(c,w)}{cw}\right) + O\left(P^{\frac{9}{4}} (sN)^{\frac{3}{4}} c^{2+\epsilon}\right). \tag{2}$$

and similarly for T_h^+ without the term coming from $h = 0$.

Evaluation of $\mathscr{B}(s)$

Putting everything together, $\mathscr{B}(s)$ splits as

- the contribution from $T(0)$ (trivial bound),
- the contribution from $T^*(c)$ (trivial bound),
- the contribution from $T^-(c) + T^+(c)$.

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- the contribution from $T^-(c) + T^+(c)$.

The contribution from $T^-(c) + T^+(c)$ splits itself as

- the contribution $T_0^-(c)$ coming from $h = 0$,
- the contribution from $Y(0)$ in $T^-(c)$
- the "Fourier transform" contribution
- the error term coming from Theorem 1.

Dealing with the error term

Subtlety: the error term $O\left(P^{\frac{9}{4}}(sN)^{\frac{3}{4}}c^{2+\epsilon}\right)$ in the evaluation of $T^{\pm}(c)$ is too weak for large c (recall $P = 1 + \sqrt{sMN}/c$).

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Use a result of Deshouiller-Iwaniec that implies the original sum $\sum_{c \equiv 0 \mod q} c^{-2} T^{\pm}(c)$ can be truncated to $c \ll C$ at a negligible cost for suitable C. So only sum the error term for $c \ll C$.

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Sum the main term in the expression (2) that we obtained for $T^{-}(c)$ (and its analogue for $T^{+}(c)$) for *all* c (not just $c \ll C$), at the price of an admissible error term.

A summation lemma

Lemma 11.1

Let f be a smooth function compactly supported on $\mathbb{R}^+$. Then

$$\sum_{c \equiv 0 \mod q} \frac{\varphi(c)}{c} f(c) = \frac{1}{\zeta(2)\nu(q)} \int f(x)dx + O\left(\frac{\varphi(q)}{q} \int |f'(x)| \log\left(1 + \frac{x}{q}\right) dx\right)$$

where

$$\nu(q) = q \prod_{p|q} \left(1 + \frac{1}{p}\right).$$

Idea: $\sum_{c \equiv 0 \mod q} \frac{\varphi(c)}{c} f(c) = \sum_d \frac{\mu(d)}{d} \sum_n f(n[d,q])$ then apply Euler-Maclaurin summation formula.

The contributions from $T_0^-(c)$ and $Y(0)$

The contribution from $T_0^-(c)$ is bounded by $q\sum_{N<n<2N}\tau(sn)\tau(n)|Q(n)|$ where

$$Q(n)=\sum_{c\equiv 0 \mod q}\frac{\varphi(c)}{c^2}G^-(sn,n)$$
$$=\int_0^\infty Y_0(4\pi t\sqrt{sn})J_{k-1}(4\pi\sqrt{sn})\sum_{c\equiv 0 \mod q}\varphi(c)F(c^2t^2,n)tdt$$

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Apply Lemma 11.1. By orthogonality of Bessel functions, get only the error term, handled by estimates for Bessel functions.

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Apply Lemma 11.1. By orthogonality of Bessel functions, get only the error term, handled by estimates for Bessel functions.
The contribution from $Y(0)$ is dealt with similarly.

End of the Proof of Theorem 2

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- $T(0)$, T_0^-, $Y(0)$ and the FT contribute $\ll \left(\frac{MN}{s}\right)^{\frac{1}{2}} q^{1+\epsilon}$. Summing the error term from Theorem 1 gives $\ll q^{\frac{3}{4}+\epsilon} s^{\frac{15}{8}} (MN)^{\frac{1}{2}}$. Adding all up gives Theorem 2 in the case $r = 1$ when s is not too big. For large s, directly use Corollary 1.

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- To relax the condition $r = 1$, take $\mathscr{F}$ to be a Hecke eigenbase and use the Hecke relations to turn $\tau(m)\tau(n)\psi_f(rm)\psi(sn)$ into a sum involving terms of the form $\tau(m')\tau(n')\psi_f(m')\psi(s'n')$ to finally get

$$\mathscr{B}(r,s) \ll q^{\epsilon}[q(r,s)(rs)^{-\frac{1}{2}} + q^{\frac{11}{12}}(rs)^{\frac{3}{4}}](MN)^{\frac{1}{2}}$$

First corolloary

From now on assume $\mathscr{F}$ is a Hecke eigenbase and denote by $\lambda_f(\ell)$ the eigenvalue of T_ℓ on the eigenfunction f.

Corollary 2

Let g be a smooth function supported on $[M, 2M]$ with $M \ll q^{1+\epsilon}$ such that $g^{(i)} \ll M^{-i}$. Let ℓ coprime with q. Then

$$\sum_{f \in \mathscr{F}} \lambda_f(\ell) \left| \sum_m \tau(m)\psi_f(m)g(m) \right|^2 \ll q^\epsilon (q\ell^{-\frac{1}{2}} + q^{\frac{11}{12}}\ell^{\frac{3}{4}})M$$

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Proof: Expand the square and use that

$$\tau(n)\lambda_f(\ell)\psi_f(n) = \sum_{\substack{a_1 a_2 n' = n \\ a_0 a_1 a_2 = \ell}} \mu(a_1)\tau(a_2)\tau(n')\psi_f(a_0 a_1 n'),$$

then use Theorem 2.

Mollification

Corollary 3

For any complex numbers c_ℓ with $(\ell, q) = 1$ we have

$$\sum_{f \in \mathscr{F}} |\Lambda_f(c)|^2 \left| \sum_m \tau(m)\psi_f(m)g(m) \right|^2 \ll q^\epsilon (q\|c\|_2^2 + q^{\frac{11}{12}} L^{\frac{3}{2}} \|c\|_1^2) M$$

where

$$\Lambda_f(c) = \sum_{l \leq L}^{*} c_\ell \lambda_f(\ell).$$

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$$\Lambda_f(c) = \sum_{l \leq L}^{*} c_\ell \lambda_f(\ell).$$

Proof: Expand $|\Lambda_f(c)|^2$ and use

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \lambda_f\left(\frac{mn}{d^2}\right)$$

then Corollary 2.

Amplification (1/2)

Corollary 4

Let $f \in \mathscr{F}$. Then we have

$$\sum_{m} \tau(m)\psi_f(m)g(m) \ll M^{\frac{1}{2}} q^{\theta+\epsilon}$$

with $\theta = \frac{47}{96}$, or $\theta = \frac{29}{60}$ if we assume

$$\sum_{l \leq L}^{*} \lambda_f^2(\ell) \gg q^{-\epsilon} L. \tag{3}$$

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Proof: Drop all but one term in Corollary 3 and make $\Lambda_f(c)$ as large as possible. If (3) holds, take $c_\ell = \lambda_f(\ell)$ and $L = q^{\frac{1}{30}}$

Amplification (2/2)

If we don't want to assume (3), use the trick $\lambda_f(p)^2 - \lambda_f(p^2) = 1$ (for p prime coprime to q) so take $c_\ell = \begin{cases} \lambda_f(\ell) \text{ if } \ell \leq L^{\frac{1}{2}} \text{ is prime,} \\ -1 \text{ if } \ell \leq L \text{ is square of a prime,} \\ 0 \text{ otherwise.} \end{cases}$

Then by the PNT $\Lambda_f(c) \sim 2L^{\frac{1}{2}}(\log L)^{-1}$.

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Then by the PNT $\Lambda_f(c) \sim 2L^{\frac{1}{2}}(\log L)^{-1}$.
By Deligne's bound (Ramanujan Conjecture) $|\lambda_f(p)| \leq 2$
$\|c\|_2^2 = \sum_{p \leq L}(1 + \lambda_f^2(p)) \leq 5\Lambda_f(c)$ and
$\|c\|_1 = \sum_{p \leq L}(1 + |\lambda_f(p)|) \leq 3\Lambda_f(c)$ so taking $L = q^{\frac{1}{24}}$ gives the result.

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Let f be a newform for $S_k(\Gamma_0(q))$ of weight $k \geq 2$.

$$L_f(s) = \sum_{n=1}^{\infty} \lambda_f(n) n^{-s},$$

where $\lambda_f(n) = \psi_f(n)/\psi_f(1)$.

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It satisfies the functional equation

$$\Psi_f(s) = i^k \epsilon_f \Psi_f(1-s) \tag{4}$$

where

$$\Psi_f(s) = \left(\frac{\sqrt{q}}{2\pi}\right)^s \Gamma\left(s + \frac{k-1}{2}\right) L_f(s)$$

and $\epsilon_f = \pm 1$ is the eigenvalue of the involution given by the action of $\left[\begin{smallmatrix} & -1 \\ q & \end{smallmatrix}\right]$ on f.

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and $\epsilon_f = \pm 1$ is the eigenvalue of the involution given by the action of $\begin{bmatrix} & -1 \\ q & \end{bmatrix}$ on f.
Convexity bound: $L_f(s) \ll q^{\frac{1}{4}} \log^2(q)$ for $\mathrm{Re}(s) = \frac{1}{2}$

Subconvexity

Theorem 3

On $\mathrm{Re}(s) = \frac{1}{2}$,

$$L_f(s) \ll q^{\frac{1}{2}\theta+\epsilon},$$

where θ is as in Corollary 4. All the derivatives $L_f^{(j)}$ satisfy the same bound.

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If we assume (3) then $\theta = \frac{1}{4} - \frac{1}{2} \times \frac{1}{60}$.
Otherwise $\theta = \frac{1}{4} - \frac{1}{2} \times \frac{1}{96}$.

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Otherwise $\theta = \frac{1}{4} - \frac{1}{2} \times \frac{1}{96}$.
Strategy of the proof: relate $L_f^2(s)$ to a Rankin-Selberg convolution $L_{\tau f}(s)$ then use Corollary 4 and an approximation argument to get an upper bound.

Relating $L_f^2(s)$ to $L_{\tau f}(s)$

If $(n,q)=1$ we have $\lambda_f(n)\psi_f(m)=\sum_{d|(m,n)}\psi_f\left(\frac{mn}{d^2}\right)$, so $\sum_{n=1}^{\infty}\psi_f(n)n^{-s}=G_f(s)H_f(s)$ where $G_f(s)=\sum_{n|q^\infty}\psi_f(n)n^{-s}$

$$H_f(s)=\sum_{(n,q)=1}\lambda_f(n)n^{-s}=\prod_{p\nmid q}(1-\lambda_f(p)p^{-s}+p^{-2s})^{-1}.$$

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If $(n,q)=1$ we have $\lambda_f(n)\psi_f(m)=\sum_{d|(m,n)}\psi_f\left(\frac{mn}{d^2}\right)$, so $\sum_{n=1}^{\infty}\psi_f(n)n^{-s}=G_f(s)H_f(s)$ where $G_f(s)=\sum_{n|q^\infty}\psi_f(n)n^{-s}$

$$H_f(s)=\sum_{(n,q)=1}\lambda_f(n)n^{-s}=\prod_{p\nmid q}(1-\lambda_f(p)p^{-s}+p^{-2s})^{-1}.$$

Same for $\tau(n)$ so $\sum_{n=1}^{\infty}\tau(n)\psi_f(n)n^{-s}=G_{\tau f}(s)H_{\tau f}(s)$,

$$H_{\tau f}(s)=\prod_{p\nmid q}(1-\lambda_f(p)p^{-s}+p^{-2s})^{-2}(1-p^{-2s})=\zeta_q(2s)^{-1}H_f^2(s).$$

Relating $L_f^2(s)$ to $L_{\tau f}(s)$

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$$\begin{aligned}\Rightarrow G_f^2(s)\zeta_q(2s)L_{\tau f}(s)&=\frac{1}{\psi_f(1)}G_f^2(s)H_f^2(s)G_{\tau f}(s)\\&=\psi_f(1)G_{\tau f}(1)L_f^2(s)\end{aligned}$$

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Relating $L_f^2(s)$ to $L_{\tau f}(s)$

If $(n,q)=1$ we have $\lambda_f(n)\psi_f(m) = \sum_{d|(m,n)} \psi_f\left(\frac{mn}{d^2}\right)$, so $\sum_{n=1}^{\infty} \psi_f(n) n^{-s} = G_f(s) H_f(s)$ where $G_f(s) = \sum_{n|q^\infty} \psi_f(n) n^{-s}$

$$H_f(s) = \sum_{(n,q)=1} \lambda_f(n) n^{-s} = \prod_{p\nmid q} (1 - \lambda_f(p)p^{-s} + p^{-2s})^{-1}.$$

Same for $\tau(n)$ so $\sum_{n=1}^{\infty} \tau(n)\psi_f(n) n^{-s} = G_{\tau f}(s) H_{\tau f}(s)$,

$$H_{\tau f}(s) = \prod_{p\nmid q} (1 - \lambda_f(p)p^{-s} + p^{-2s})^{-2}(1 - p^{-2s}) = \zeta_q(2s)^{-1} H_f^2(s).$$

$$\Rightarrow G_f^2(s)\zeta_q(2s)L_{\tau f}(s) = \frac{1}{\psi_f(1)} G_f^2(s) H_f^2(s) G_{\tau f}(s)$$
$$= \psi_f(1) G_{\tau f}(1) L_f^2(s)$$

f newform implies $G_f^2(s) = \psi_f(1) G_{\tau f}(s)$ hence

$$L_f^2(s) = \zeta_q(2s) L_{\tau f}(s).$$

Estimate for truncated sums

Write $\psi_f(1)L_f^2(s) = \sum_{n=1}^{\infty} \rho_f(n) n^{-s}$. Since $L_f^2(s) = \zeta_q(2s) L_{\tau f}(s)$ if follows

$$\rho_f(n) = \sum_{\substack{d^2 m = n \\ (d,q)=1}} \tau(m)\psi_f(m).$$

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Hence by Corollary 4 if g is a smooth function supported on $[X, 2X]$ with $X \ll q^{1+\epsilon}$ satisfying $g^{(j)} \ll X^{-j}$ then

$$S(g) \doteq \sum_{n=1}^{\infty} \rho_f(n)n^{-\frac{1}{2}}g(n) \ll q^{\theta+\epsilon}. \tag{5}$$

Removing the restriction on X

By Mellin inversion, the functional equation (4) and shifting the contour

$$\begin{aligned} S(g) &= \frac{1}{2i\pi}\sum_{n=1}^{\infty}\rho_f(n)n^{-\frac{1}{2}}\int_{\mathrm{Re}(s)=1}\hat{g}(s)n^{-s}ds \\ &= \psi_f(1)\frac{1}{2i\pi}\int_{\mathrm{Re}(s)=1}L_f^2\left(s+\frac{1}{2}\right)\hat{g}(s)ds \\ &= \psi_f(1)\frac{1}{2i\pi}\int_{\mathrm{Re}(s)=1}L_f^2\left(\frac{1}{2}-s\right)\left(\frac{4\pi^2}{q}\right)^{2s}\frac{\Gamma^2(\frac{k}{2}-s)}{\Gamma^2(\frac{k}{2}+s)}\hat{g}(s)ds \\ &= S(g^*), \end{aligned}$$

where $g^*(t) = h\left(\frac{16\pi^4}{q^2}t\right)$ and $h(y) = \int_{\mathrm{Re}(s)=0}\frac{\Gamma^2(\frac{k}{2}-s)}{\Gamma^2(\frac{k}{2}+s)}\hat{g}(s)y^s ds$ has rapid decay in the range $y \gg X^{-1}$ so (5) holds without the restriction on X.

Conclusion of the proof

Partition $L_f^2(s)$ into sums of the type $\psi_f(1)S(g)$. Estimate (5) gives

$$L_f^2(s) \ll \psi_f(1)^{-1} q^{\theta+\epsilon}.$$

The bound $\psi_f(1) \gg q^{-\epsilon}$ is known. Finally to prove the statement for the derivatives $L_f^{(j)}(s)$, replace $g(n)$ with $g(n)(\log n)^j$.