GL(3) subconvexity in the twist aspect

Yongxiao Lin
EPF Lausanne

London learning seminar on analytic number theory
March 24, 2021

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Self-dual case: a result of Blomer

Let χ mod q be a primitive Dirichlet character. Let $\varphi \in \mathrm{GL}_3(\mathbb{Z})$ be a *fixed* Hecke–Maass cusp form.
Using a first moment method (à la Xiaoqing Li)

$$\frac{1}{q} \sum_{f_j \in \mathcal{B}(q)} (1+|t_j|)^{-B} L(1/2, \varphi \times f_j.\chi) + \frac{1}{q} \int_{-\infty}^{\infty} \frac{|t|^2}{(1+|t|)^B} |L(1/2+it, \varphi.\chi)|^2 \mathrm{d}t \ll q^{1/4+\varepsilon},$$

Blomer derived (among other thing)

$$L(1/2, \varphi.\chi_q) \ll_\varphi q^{5/8+\varepsilon},$$

for $\varphi = \mathrm{sym}^2 g$ and $\chi_q^2 = 1$.
Question: How to extend the results to more general φ and χ?

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Why the t-aspect approach fails?

Recall in his [III], Munshi used Kloosterman's circle method + "conductor-lowering" trick (removed by Aggarwal later)

$$\frac{1}{K}\int_{\mathbb{R}} V\Big(\frac{v}{K}\Big)\Big(\frac{n}{r}\Big)^{iv}\mathrm{d}v \approx 1_{|n-r|<N/K}$$

to obtain $L(1/2+it,\varphi) \ll_{\varphi} t^{3/4-\delta}$. (See lecture by Jesse Jääsaari.) Such a feature does not exist in the q-aspect case, i.e., we would need

$$\sum_{n\sim N}\lambda_{\varphi}(1,n)\chi(n) = \sum_{n\sim N}\lambda_{\varphi}(1,n)\sum_{r\sim N}\chi(r)\delta\Big(\frac{n-r}{\ell},0\Big)\cdot 1_{\ell|(n-r)}$$

for some $\ell|q$ and then apply DFI to detect $\delta\big((n-r)/\ell,0\big)$. This is possible only if the moduli q are composite (e.g., $q=q_1q_2$ or $q=p^r$).

It was unclear how to extend the t-aspect approach to the q-aspect case (with q prime).

Character twists of GL_3: Munshi's result

Munshi took a different approach and showed

Theorem (Munshi, 15'-Annals +16'-preprint)

Let φ be a $SL_3(\mathbb{Z})$-invariant cusp form. Let q be a prime. Let $\chi \pmod q$ be primitive Dirichlet characters. Then

$$L(1/2, \varphi.\chi) \ll_{\varphi} q^{3/4-\delta+o(1)}, \ \delta = 1/308.$$

Munshi's proof uses

- a variant of the DFI δ-symbol method (via Petersson trace formula), Voronoi summation, reciprocity for Kloosterman fractions ($e(\frac{n\bar{m}}{q}) = e(-\frac{n\bar{q}}{m})e(\frac{n}{qm})$), and Cauchy–Schwarz.
- multiplicativity of Dirichlet characters, so that "amplification" is possible.

Later, Holowinsky and Nelson found a simplification of Munshi's approach, leading to:

Character twists of GL_3: HN's improvement

Theorem (Holowinsky–Nelson, 2018)

Same notation as above. Then

$$L(1/2, \varphi.\chi) \ll_\varphi q^{3/4-\delta+o(1)}, \ \delta = 1/36.$$

- Their proof removed the use of the Petersson δ-symbol.
- This was extended to general trace functions K by Kowalski–L.–Michel–Sawin, who showed that

$$\sum_{n\geq 1} \lambda_\varphi(1,n) K(n) V\left(\frac{n}{N}\right) = o(N)$$

  as long as $N \gg q^{4/3}$ and $\|\widehat{K}\|_\infty \ll 1$. Here

$$\widehat{K}(z) = \frac{1}{q^{1/2}} \sum_{x(q)} K(x) e\left(\frac{xz}{q}\right)$$

  denotes the (discrete) Fourier transform of K.

Below we sketch Munshi's approach and HN's simplification.

proof preparation: Poisson & Voronoi summations

Let $K(n;q)$ be a q-periodic function.

Lemma (Poisson summation: GL_1)

$$\sum_{n\sim N} K(n;q) \approx \frac{N}{\sqrt{q}} \sum_{\tilde{n}\ll \frac{q}{N}} \widehat{K}(\tilde{n};q),$$

where $\widehat{K}(\tilde{n};q) = \frac{1}{\sqrt{q}} \sum_{x(q)} K(x;q) e\left(\frac{x\tilde{n}}{q}\right)$.

Let

$$Kl_i(m;q) := \frac{1}{q^{\frac{i-1}{2}}} \sum_{x_1,\cdots,x_{i-1}(q)} e\Big(\frac{x_1+\cdots+m\,\overline{x_1x_2\cdots x_{i-1}}}{q}\Big)$$

be $(i-1)$-dim'l normalized hyper-Kloosterman sums.

Lemma (Voronoi summation: $\mathrm{GL}_\mathrm{d}(\mathbb{Z})$-case)

$$\sum_{n\sim N} \frac{\lambda_F(n) Kl_i(an;q)}{\sqrt{n}} \approx \sum_{\tilde{n}\ll \frac{q^d}{N}} \frac{\overline{\lambda_F(\tilde{n})} Kl_{d-i}(\bar{a}\tilde{n};q)}{\sqrt{\tilde{n}}}.$$

Recap of versions of delta-symbol

For $-N \leq n \leq N$, to detect $\delta(n,0)$, we have

- "trival" δ-symbol

$$\delta(n,0) = \frac{1}{c}\sum_{a(c)} e\Big(\frac{an}{c}\Big), \quad \text{if } c > |n|;$$

application: subconvexity for $L(1/2+it, f.\chi)$ (Burgess-Weyl type), $L(1/2, f \times g)$, etc.

- Jutila δ-symbol

$$\delta(n,0) \approx \frac{1}{|\mathcal{C}|^2}\sum_{c\in\mathcal{C}} \sum_{a(c)}^{\star} e\Big(\frac{an}{c}\Big)\frac{1}{2\delta}\int_{-\delta}^{\delta} e(nx)\mathrm{d}x + \text{"error"};$$

application: subconvexity for $L(1/2+it, f.\chi)$, $\mathrm{GL}_3 \times \mathrm{GL}_2$ shifted convolution.

Recap of versions of delta-symbol (cont)

- Kloosterman δ-symbol

$$\delta(n,0) = 2\mathrm{Re} \sum_{c \le C < a \le c+C} \sum{}^{\star} \frac{1}{ca} e\left(\frac{\bar{a}n}{c}\right) \int_0^1 e\left(-\frac{nx}{ca}\right) \mathrm{d}x;$$

  application: quaternary quadratic forms, subconvexity for $L(1/2+it,\varphi)$, etc.

- DFI δ-symbol

$$\delta(n,0) = \frac{1}{C} \sum_{1 \le c \le C} \frac{1}{c} \sum_{a(c)}{}^{\star} \underbrace{e\left(\frac{an}{c}\right)}_{arithmetic} \int_{\mathbb{R}} g(c,x) \underbrace{e\left(\frac{nx}{cC}\right)}_{archimedean} \mathrm{d}x;$$

  various of applications...

and other versions...

proof preparation: Petersson δ-symbol

Iwaniec (97', "Topics") interpreted the Petersson formula

$$\sum_{f\in\mathcal{B}_k(m,\psi)} \omega_f^{-1}\overline{\lambda_f(r)}\lambda_f(n) = \delta(r,n) + 2\pi i^{-k}\sum_{c=1}^{\infty}\frac{S_\psi(r,n;cm)}{cm}J_{k-1}\left(\frac{4\pi\sqrt{rn}}{cm}\right)$$

as a spectral decomposition of $\delta(r,n)$, and performed averaging over the weight k to derive applications.

Here

$$S_\psi(r,n;c) = \sum_{x \bmod c}^{\star}\psi(x)e\left(\frac{rx+n\bar{x}}{c}\right).$$

Munshi wrote

$$\delta(r,n) = \sum_{f\in\mathcal{B}_k(m,\psi)} \omega_f^{-1}\overline{\lambda_f(r)}\lambda_f(n) - 2\pi i^{-k}\sum_{c=1}^{\infty}\frac{S_\psi(r,n;cm)}{cm}J_{k-1}\left(\frac{4\pi\sqrt{rn}}{cm}\right).$$

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Averaging over Petersson formula

For χ mod q, to show

$$L(1/2, \varphi.\chi) \ll_\varphi q^{3/4-\delta},$$

By *Approx. FE*, it suffices to show

$$\sum_{n \sim N} \lambda_\varphi(1,n)\chi(n) \ll_\varphi N^{1-\delta'}, \quad N \ll q^{3/2}.$$

Munshi (16'-preprint) took the following averages over ψ and m $(= pq)$:

$$\delta(r, n\ell) = \frac{1}{P^\star} \sum_{p \sim P} \sum_{\substack{\psi \bmod p \\ \psi(-1)=-1}} \sum_{f \in \mathcal{B}_k(pq,\psi)} \omega_f^{-1} \overline{\lambda_f(r)} \lambda_f(n\ell)$$

$$- \frac{2\pi i^{-k}}{P^\star} \sum_{p \sim P} \sum_{\substack{\psi \bmod p \\ \psi(-1)=-1}} \sum_{c=1}^{\infty} \frac{S_\psi(r, n\ell; cpq)}{cpq} J_{k-1}\left(\frac{4\pi\sqrt{rn\ell}}{cpq}\right).$$

Here $P^\star \approx P^2$.

Munshi's approach: applying Petersson δ-symbol

With this expression, Munshi wrote (assume $N = q^{3/2}$)

$$\sum_{n\sim q^{3/2}} \lambda_\varphi(1,n)\chi(n) \approx \frac{1}{L}\sum_{\ell\sim L}\bar{\chi}(\ell) \sum_{n\sim q^{3/2}} \lambda_\varphi(1,n) \sum_{r\sim q^{3/2}L} \chi(r)\,\delta(r,n\ell)$$
$$:= \mathcal{F}^\star + \mathcal{O}^\star.$$

Rmk: the ℓ-sum is reminiscent of the amplification method; multiplicity of χ: $\chi(n\ell) = \chi(n)\chi(\ell)$ is needed.

The Goal: To beat the bounds $\mathcal{F}^\star, \mathcal{O}^\star \ll q^{3/2}$ (w/ appropriate choices of P and L). Here

$$\mathcal{O}^\star \approx \frac{1}{P^2L}\sum_{p\sim P}\sum_{\ell\sim L}\bar{\chi}(\ell) \sum_{n\sim q^{3/2}} \lambda_\varphi(1,n) \sum_{r\sim q^{3/2}L} \chi(r)$$
$$\sum_{\substack{\psi \bmod p \\ \psi(-1)=-1}} \sum_{c\ll\sqrt{q}L/P} \frac{S_\psi(r,n\ell;cpq)}{cpq} J_{k-1}\left(\frac{4\pi\sqrt{rn\ell}}{cpq}\right),$$

and $\mathcal{F}^\star =$ (next slide).

Munshi's approach: treatment of $\mathcal{F}^\star$

$$\mathcal{F}^\star \approx \frac{1}{P^2 L} \sum_{p\sim P} \sum_{\ell \sim L} \bar{\chi}(\ell) \sum_{\substack{\psi \bmod p \\ \psi(-1)=-1}} \sum_{f\in\mathcal{B}_k(pq,\psi)} \omega_f^{-1}$$

$$\sum_{r\sim q^{3/2}L} \chi(r)\overline{\lambda_f(r)} \sum_{n\sim q^{3/2}} \lambda_\varphi(1,n)\lambda_f(n\ell).$$

Munshi took the following steps to treat $\mathcal{F}^\star$.

- Functional equations to the n and r-sums.
- Petersson formula over f. No diagonal contribution; only left with the off-diagonal terms $\sum_{c\ll\sqrt{q}P^2}$. Sum over ψ and simplify the sum.
- GL_3-Voronoi over the n-sum with modulus c.
- Poisson summation over the c-sum to arrive at

$$\sum_{p\sim P} \sum_{\ell\sim L} \sum_{r\sim\sqrt{q}P/L} \sum_{n\sim P^3} \lambda_\varphi(1,n)\bar{\chi}(r\ell\bar{p})S(-qn\bar{p},1;r\ell).$$

- Cauchy–Schwarz with p,ℓ-sums inside the square $\sum_n \sum_r \left|\sum_p \sum_\ell \bar{\chi}(\ell\bar{p})S(-qn\bar{p},1;r\ell)\right|^2$, followed with Poisson in

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Holowinsky's initial alternative treatment of $\mathcal{F}^\star$

- Functional equation to the r-sum only.
- Petersson formula over f. Simplifying the sum gives

$$\sum_{p\sim P}\sum_{\ell\sim L}\sum_{r\sim\sqrt{q}P/L}\sum_{n\sim q^{3/2}}\lambda_\varphi(1,n)\bar{\chi}(r\ell\bar{p})S(nr\ell\bar{p},1;q).$$

- GL_3-Voronoi over the n-sum to get

$$\mathcal{F}^q=\frac{1}{P^2}\sum_{p\sim P}\sum_{\ell\sim L}\sum_{r\sim\sqrt{q}P/L}\sum_{n\sim q^{3/2}}\lambda_\varphi(1,n)\bar{\chi}(r\ell\bar{p})e\left(\frac{n\overline{r\ell}p}{q}\right).$$

- Reciprocity $e\left(\frac{n\overline{r\ell}p}{q}\right)=e\left(-\frac{np\bar{q}}{r\ell}\right)e\left(\frac{np}{r\ell q}\right)$, and then Voronoi in the n-sum with modulus $r\ell$, to arrive at

$$\sum_{p\sim P}\sum_{\ell\sim L}\sum_{r\sim\sqrt{q}P/L}\sum_{n\sim P^3}\lambda_\varphi(1,n)\bar{\chi}(r\ell\bar{p})S(-qn\bar{p},1;r\ell).$$

- Same as in Munshi's.

Holowinsky–Nelson's observation

HN observed: if one applied Poisson summation over the r-sum in $\mathcal{F}^q$, the zero frequency $\tilde{r} = 0$ contributes our initial object of interest $\sum_{n\sim q^{3/2}} \lambda_\varphi(1,n)\chi(n)$, and the non-zero frequencies $\tilde{r} \neq 0$ give a sum $\mathcal{O}^q$ (similar to Munshi's $\mathcal{O}^\star$-term).
i.e., they discovered the relation

$$\begin{aligned}\mathcal{F}^q =& \frac{1}{P^2} \sum_{p\sim P} \sum_{\ell\sim L} \sum_{r\sim\sqrt{q}P/L} \sum_{n\sim q^{3/2}} \lambda_\varphi(1,n)\chi(r\ell\bar{p}) e\left(\frac{n\overline{r\ell}p}{q}\right) \\ &\approx \varepsilon_{\bar{\chi}} \sum_{n\sim q^{3/2}} \lambda_\varphi(1,n)\chi(n) + \mathcal{O}^q,\end{aligned}$$

where

$$\mathcal{O}^q = \frac{1}{PL} \sum_{n\sim q^{3/2}} \lambda_\varphi(1,n) \sum_{p\sim P} \sum_{\ell\sim L} \sum_{0\neq|\tilde{r}|\ll\sqrt{q}L/P} \frac{S_{\bar{\chi}}(n, \tilde{r}p\bar{\ell}; q)}{\sqrt{q}}.$$

$\rightsquigarrow$ one can eliminate Munshi's previous steps (particularly the Petersson δ-symbol), and begin just with the sum $\mathcal{F}^q$.

One-page summary of HN's simplification ($\chi \rightsquigarrow K \bmod q$)

Poisson summation gives

$$\frac{L}{P}\sum_{r\sim\sqrt{q}P/L}\widehat{K}(-\overline{r\ell}p)e\Big(\frac{\overline{r\ell}np}{q}\Big)=K(n)+\sum_{0\neq\tilde{r}<\sqrt{q}L/P}\widetilde{S}_{\widehat{K}}(n,\tilde{r}p\bar{\ell};q),$$

where $\widetilde{S}_{\widehat{K}}(n,\tilde{r}p\bar{\ell};q):=\frac{1}{\sqrt{q}}\sum_{z(q)}\widehat{K}(z)e\big(\frac{-zn}{q}\big)e\big(\frac{-\bar{z}\tilde{r}p\bar{\ell}}{q}\big)$. Recall here $\widehat{K}(z)=\frac{1}{q^{1/2}}\sum_{x(q)}K(x)e\big(\frac{zx}{q}\big)$.

Then

$$\mathcal{F}_q=\sum_{n\sim q^{3/2}}\lambda_\varphi(1,n)K(n)+\mathcal{O}_q,$$

where

$$\mathcal{F}_q:=\frac{1}{P^2}\sum_{p\sim P}\sum_{\ell\sim L}\sum_{r\sim\sqrt{q}P/L}\sum_{n\sim q^{3/2}}\lambda_\varphi(1,n)\widehat{K}(-\overline{r\ell}p)e\Big(\frac{\overline{r\ell}pn}{q}\Big);$$

and

$$\mathcal{O}_q:=\frac{1}{PL}\sum_{n\sim q^{3/2}}\lambda_\varphi(1,n)\sum_{p\sim P}\sum_{\ell\sim L}\sum_{0\neq\tilde{r}<\sqrt{q}L/P}\widetilde{S}_{\widehat{K}}(n,\tilde{r}p\bar{\ell};q).$$

Munshi and HN's treatment for $\mathcal{F}_q$ and $\mathcal{O}_q$

- Treatment of $\mathcal{O}_q$: Cauchy–Schwarz

$$\mathcal{O}_q \ll \frac{1}{PL}\Big(\sum_{n\sim q^{3/2}} |\lambda_\varphi(1,n)|^2\Big)^{1/2}\Big(\sum_{n\sim q^{3/2}} \Big|\sum_p \sum_\ell \sum_{\tilde r\neq 0} \widetilde{S}_{\widehat{K}}(n, \tilde r p \bar\ell; q)\Big|^2\Big)^{1/2};$$

Open the square and Poisson in the n-variable $\ll q^{3/2}\|\widehat{K}\|_\infty \frac{q^{1/4}}{P}$.

- Treatment of $\mathcal{F}_q$: Reciprocity $e(\frac{\overline{r\ell} np}{q}) = e(-\frac{np\bar q}{r\ell})e(\frac{np}{r\ell q})$, then Voronoi $\sum_n \lambda_\varphi(1,n)e(-\frac{np\bar q}{r\ell}) \to \sum_n \overline{\lambda_\varphi(1,n)}\mathrm{Kl}_2(\bar p q n; r\ell)$, to get

$$\begin{aligned}\mathcal{F}_q =& \frac{q^{3/4}}{P^{7/2}} \sum_{n\sim P^3} \sum_{r\sim\sqrt{q}P/L} \overline{\lambda_\varphi(1,n)} \sum_{p\sim P} \sum_{\ell\sim L} \widehat{K}(-\overline{r\ell}p)\mathrm{Kl}_2(\bar p q n; r\ell) \\ \ll& \frac{q^{3/4}}{P^{7/2}}\Big(\sum_{n\sim P^3}\sum_r |\overline{\lambda_\varphi(1,n)}|^2\Big)^{1/2}\Big(\sum_{n\sim P^3}\sum_r \Big|\sum_p\sum_\ell (\cdots)\Big|^2\Big)^{1/2}.\end{aligned}$$

Open the square and apply Poisson in the n-sum,
$\ll q^{3/2}\|\widehat{K}\|_\infty(\frac{P}{q^{1/4}L^{1/2}} + (\frac{PL}{q^{1/2}})^{1/4})$.

- Balance the parameters $\Rightarrow \mathcal{F}_q + \mathcal{O}_q \ll \|\widehat{K}\|_\infty q^{3/2-\delta}$; done.

Comparisons: character twists of $\mathrm{GL_d}$ $(d = 1, 2, 3 \times 2)$

Question: What the shortest range is for getting cancellation? i.e., how small η can we take so that

$$\sum_{n \sim (\sqrt{\mathrm{Cond}})^{\eta}} \lambda_F(n) K(n) = o(N)?$$

- GL_1 case. Pólya–Vinogradov: $\sum_{n<N} \chi(n) \ll \sqrt{q} \log q$; Burgess: $\sum_{n<N} \chi(n)$ has cancellation when $N \gg q^{1/4}$.
- GL_2 case. Fouvry–Kowalski–Michel (aprés Bykovskiĭ):

$$\sum_{n \sim N} \lambda_f(n) K(n) V\left(\frac{n}{N}\right) \ll_{f,V,C(\rho)} N^{1/2} q^{3/8},$$

  hence it has cancellation when $N \gg q^{3/4}$.
- $\mathrm{GL}_3 \times \mathrm{GL}_2$ case. Sharma and L.–Michel–Sawin show

$$\sum_{r,n \geq 1} \lambda_\varphi(r,n) \lambda_f(n) K(n) V\left(\frac{r^2 n}{N}\right) = o(N),$$

  when $N \gg q^{3-1/4} = (q^3)^{\frac{11}{12}}$.

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“natural” threshold for shortest length of cancellation?

- **Guess:** Maybe a “natural” proof in all these problems would produce an exponent $\frac{2d-1}{2d}$ (s.t. $N \gg (\sqrt{\text{Cond}})^{\frac{2d-1}{2d}}$)?

  This holds for $d = 1, 2, 3 \times 2$.

- *Issue*: The $d = 3$ case does not match this bound yet: a *“natural”* proof should give $N \gg q^{5/4} = (q^{\frac{3}{2}})^{\frac{5}{6}}$.

  Recall Holowinsky–Nelson and KLMS obtain

  $$\sum_{n \geq 1} \lambda_\varphi(1, n) K(n) V\left(\frac{n}{N}\right) \ll_{\varphi, V} q^{2/9} N^{5/6} \|\widehat{K}\|_\infty,$$

  showing cancellation for $N \gg q^{4/3}$. Though this is sufficient for subconvexity, for other applications it is desirable to go down further. **Example:** If one was able to obtain cancellation for $N \gg q$, then taking $K(n) = Kl_3(an; q)$ this would imply $\{\lambda_\varphi(1, n) : n \leq X\}$ being equidistributed in $n \equiv a \bmod q$, for $q < X^{\vartheta_3}$ with $\vartheta_3 = 1/2 + \eta$.

Limitation of the Munshi-HN approach

- By specifying Sharma's $f \in \mathrm{GL}_2$ to Eisenstein series, one gets:

$$\sum_{n \geq 1} \lambda_\varphi(1,n)\chi(n)V\Big(\frac{n}{N}\Big) \ll N^{1-\delta'},$$

when $N \gg q^{11/8} = (q^{\frac{3}{2}})^{\frac{11}{12}}$, improvement over Holowinsky–Nelson's $N \gg q^{4/3}$ (and also the subconvex-exponent).

- The approach of Munshi and HN does not make heavy use of the underlying geometry of χ (e.g., Weil's *RH*). Instead of encountering

$$\sum_{x \bmod q} S_{\bar{\chi}}(x, \tilde{r}p_1\bar{\ell}_1; q)\overline{S_{\bar{\chi}}(x, \tilde{r}p_2\bar{\ell}_2; q)}\, e\Big(\frac{x\tilde{n}}{q}\Big)$$

with $\tilde{n} \neq 0$, one only needs to deal with the case $\tilde{n} = 0$, in contrast to the GL_2 and $\mathrm{GL}_3 \times \mathrm{GL}_2$-scenario.

Problem: Can one find a new proof for $\sum_{n \sim N} \lambda_\varphi(1,n)K(n) = o(N)$ that shows cancellation for $N > q^{5/4}$? Or at least improving $N > q^{4/3}$?

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$$\sum_{n \geq 1} \lambda_\varphi(1,n)\chi(n)V\Big(\frac{n}{N}\Big) \ll N^{1-\delta'},$$

when $N \gg q^{11/8} = (q^{\frac{3}{2}})^{\frac{11}{12}}$, improvement over Holowinsky–Nelson's $N \gg q^{4/3}$ (and also the subconvex-exponent).

- The approach of Munshi and HN does not make heavy use of the underlying geometry of χ (e.g., Weil's *RH*). Instead of encountering

$$\sum_{x \bmod q} S_{\bar{\chi}}(x, \tilde{r}p_1\bar{\ell}_1; q)\overline{S_{\bar{\chi}}(x, \tilde{r}p_2\bar{\ell}_2; q)}\, e\Big(\frac{x\tilde{n}}{q}\Big)$$

with $\tilde{n} \neq 0$, one only needs to deal with the case $\tilde{n} = 0$, in contrast to the GL_2 and $\mathrm{GL}_3 \times \mathrm{GL}_2$-scenario.

Problem: Can one find a new proof for $\sum_{n \sim N} \lambda_\varphi(1,n)K(n) = o(N)$ that shows cancellation for $N > q^{5/4}$? Or at least improving $N > q^{4/3}$?

Limitation of the Munshi-HN approach

Examples with better saving.

- In the self-dual case, Blomer proved much stronger bounds

$$L(1/2, \mathrm{sym}^2 g.\chi_q) \ll q^{3/4-1/8+\varepsilon}, \quad L(1/2, \mathrm{sym}^2 g \times f_j.\chi_q) \ll q^{3/2-1/4+\varepsilon}.$$

- If $q = p^r$, p fixed prime and r large, then Sun–Zhao:

$$\sum_{n \sim N} \lambda_\varphi(1, n)\chi(n) \ll_{p,\varphi} N^{1/2} q^{3/4-3/40},$$

showing cancellation for $N > (q^{\frac{3}{2}})^{\frac{9}{10}}$, improving prime q case.

- In the t-aspect case, Aggarwal (improving Munshi's [III]):

$$\sum_{n \sim N} \lambda_\varphi(1, n) n^{it} \ll_\varphi N^{3/4} t^{3/10},$$

showing cancellation for $N > t^{\frac{6}{5}}$ (note: $\frac{6}{5} < \frac{5}{4}$).

But for prime q, this remains unclear...

Why one cares about cancellation in shorter ranges?

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Level of distribution: finer question beyond subconvexity

"*level of distribution*" ϑ_d for $\mathrm{GL_d}$ Hecke eigenvalues $\{\lambda_F(n) : n \le X\}$ in arithmetic progressions $n \equiv a \bmod q$:

$$\sum_{\substack{n \le X \\ n \equiv a(q)}} \lambda_F(n) - \frac{1}{\varphi(q)} \sum_{\substack{n \le X \\ (n,q)=1}} \lambda_F(n) \ll_{F,A} \frac{X}{q}\big(\log X\big)^{-A},$$

for $q \le X^{\vartheta_d}$.
By applying functional eq./Voronoi, one can take $\vartheta_d = \frac{2}{d+1} - \varepsilon$.
GRH $\Rightarrow \vartheta_d = 1/2 - \varepsilon$.
To improve $\vartheta_d = \frac{2}{d+1} - \varepsilon$ for $F \in \mathrm{GL_d}$, one would need

$$\sum_{n \sim N} \lambda_F(n) \mathrm{Kl}_d(an; q) V\big(\frac{n}{N}\big) = o(N)$$

for $N \approx q^{\frac{d}{2} - \frac{1}{2}}$. However, this seems only known when F are certain Eisenstein series (e.g., $\lambda_F(n) = \sum_{m|n} \lambda_f(m), \tau_d(n)$, etc).

Some other questions

- How to extend Conrey–Iwaniec and Petrow–Young's *Weyl* bound results to trace functions K mod q:

$$\sum_{n\geq 1}\lambda_f(n)K(n)V\big(\frac{n}{q}\big) \ll_f q^{1-\delta+o(1)},\ \delta = 1/6?$$

(Maybe for special trace functions (e.g. $K(n) = e(\frac{\overline{n}}{q})$) first?)

- Will it be possible (by "shortening" the family) to establish *sub*-Weyl for twists of GL_2 for composite moduli ($q = q_1q_2$ or $q = p^r$, say)

$$L(1/2, f.\chi) \ll_f q^{1/3-\delta}?$$

- Adelize the delta symbols, so that these results can be extended to number fields?

Leave it to you!

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