Moments and Amplification I

Radu Toma

Mathematisches Institut, Universität Bonn

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GL_2 L-functions

Let

$$f(z) = \sum_n a_f(n) e(nz)$$

be a cusp form of weight k for the congruence subgroup $\Gamma_0(q)$ with norm 1.
Analytic number theory normalisation:

$$\psi_f(n) = \left(\frac{q(k-1)!}{(4\pi n)^{k-1}} \right)^{1/2} a_f(n)$$

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Suppose f is a newform. Then

$$\lambda(n) = \psi_f(n)/\psi_f(1)$$

is the (normalised) Hecke eigenvalue if $(n, q) = 1$.

GL_2 L-functions

Define

$$L_f(s) = \sum_{n=1}^{\infty} \lambda_f(n) n^{-s}.$$

A newform f is an eigenfunction of the Fricke involution $Wf(z) = f|\begin{pmatrix} & -1 \\ q & \end{pmatrix}$.
Suppose that

$$Wf = \varepsilon_f f, \ \varepsilon_f = \pm 1.$$

Then we have the functional equation

$$\Lambda_f(s) = i^k \varepsilon_f \Lambda_f(1-s),$$

where

$$\Lambda_f(s) = q^{s/2}(2\pi)^{-s}\Gamma(s+(k-1)/2)L_f(s).$$

We have the Euler product

$$L_f(s) = \prod_p \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{\chi_0(p)}{p^{2s}}\right)^{-1}$$

Subconvexity. The goal and the methods

The convexity bound is

$$L_f(s) \ll q^{1/4} log^2(q),$$

for $\Re s = 1/2$.

Theorem (Duke, Friedlander, Iwaniec)

We have the bound

$$L_f(1/2 + it) \ll q^{1/4 - 1/192 + \varepsilon}.$$

Methods for proving: moments and amplification + lots of other (deep) ingredients.

Method of moments. General remarks

Let f_0 be our chosen form for which we want to prove subconvexity. Observe that

$$|L_{f_0}(1/2)|^2 = \sum_{n,m} \frac{\lambda_{f_0}(n)\overline{\lambda_{f_0}(m)}}{(nm)^{1/2}},$$

and in general the absolute value to the $2k$-th power is made up of sums of products of Hecke eigenvalues.

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and in general the absolute value to the $2k$-th power is made up of sums of products of Hecke eigenvalues.
Recall the Petersson formula

$$C_k(nm)^{(1-k)/2} \sum_{f \in \text{ONB}} \overline{a_f(n)} a_f(m)$$
$$= \delta_{mn} + C'_k \sum_{q|c} \frac{S(n,m,c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)$$

Method of moments. General remarks

Hope: using harmonic analysis (e.g. Petersson, Kuznetsov) could prove the generalised Lindelöf hypothesis *on average* over a *nice* family of forms $\mathcal{F}$, i.e.

$$\frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} |L_f(1/2)|^{2k} \ll q^{\epsilon}.$$

If $f_0 \in \mathcal{F}$, by positivity we get

$$|L_{f_0}(1/2)| \ll |\mathcal{F}|^{\frac{1}{2k}} q^{\epsilon},$$

and we win if

$$|\mathcal{F}|^{\frac{1}{2k}} \ll q^{1/4-\delta}.$$

In a tight situation: amplification

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Sometimes we can prove Lindelöf on average, but barely miss subconvexity.
Taking a higher moment would be much more difficult.
Iwaniec's solution is **amplification**: make use of the extra symmetries of the family (Hecke operators) to amplify the contribution of L_{f_0} and break convexity.

Amplification. General remarks

We introduce an extra shorter linear form

$$A(f) = \sum_{l \leq L} c_l \lambda_f(l),$$

for some parameter L usually a small power of q. We hope that

$$\frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} |L_f(1/2)|^{2k} |A(f)|^2 \ll q^{\varepsilon} \sum |c_l|^2.$$

Then we might (try to) construct $A(f)$ so that

$$\sum_{l \leq L} |c_l|^2 \ll q^{\varepsilon} L^{\alpha+\varepsilon} \quad \text{and} \quad |A(f_0)| \gg q^{-\varepsilon} L^{\alpha-\varepsilon},$$

for some $\alpha > 0$. Then we prove subconvexity.

Constructing the amplifier

In previous talks, we had sums $\sum_{\chi}^{*}$ and the amplifier was $A(\chi) = \sum_{l} c_l \chi(l)$. Using orthogonality of characters we had a great amplifier

$$A(\chi) = \sum_{l} \overline{\chi_0(l)} \chi(l),$$

with $\alpha = 1$.
In our case, the natural choice $c_l = \overline{\lambda_{f_0}(l)}$ creates difficulties in the general. Iwaniec made clever use of the Hecke relation

$$\lambda_f(p)^2 - \lambda_f(p^2) = 1$$

to bypass these and obtained an amplifier with $\alpha = 1/2$. Watch this in action in the next episode of *Moments and amplification*, presented by Félicien.

Hecke relations. Towards the fourth moment

Hecke eigenvalues are multiplicative:

$$\lambda(m)\lambda(n) = \sum_{d|(m,n)} \lambda\left(\frac{mn}{d^2}\right),$$

for $(mn, q) = 1$. This is how we get the Euler product

$$U_f(s) = \sum_{(n,q)=1} \lambda_f(n) n^{-s} = \prod_{p\nmid q} (1 - \lambda_f(p)p^{-s} + p^{-2s})^{-1}.$$

Tempted by the method of moments, we note that

$$\zeta_q(2s)^{-1} U_f(s)^2 = \prod_{p\nmid q} (1 - \lambda_f(p)p^{-s} + p^{-2s})^{-2}(1 - p^{-2s})$$
$$= \sum_{(n,q)=1} \tau(n)\lambda_f(n) n^{-s} =: U_{\tau f}(s).$$

Calculus. Rankin-Selberg convolution

Where does this come from? For instance, say $X = p^{-s}$ so that the Euler factor of $U_{\tau f}$ is

$$\sum_k (k+1)\lambda(p^k)X^k,$$

recalling that $\tau(p^k) = k+1$ and τ is multiplicative. Then the above is

$$\left(\sum_k \lambda(p^k)X^{k+1}\right)' = \left(\frac{X}{1-\lambda(p)X+X^2}\right)' = \frac{1-X^2}{(1-\lambda(p)X+X^2)^2}.$$

A good way to think about this: $U_{\tau f}$ is the Rankin-Selberg convolution of f with an Eisenstein series. More precisely, we have

$$\frac{\partial}{\partial s}E(z,s)|_{s=1/2} = y^{1/2}\log y + 4y^{1/2}\sum_1^\infty \tau(n)K_0(2\pi ny)\cos(2\pi nx).$$

The fourth moment

Taking f a newform we can deal with the ramified places and then

$$L_f^2(s) = \zeta_q(2s) L_{\tau f}(s), \quad L_{\tau f}(s) = \sum_n \tau(n)\psi_f(n) n^{-s}.$$

Bounding the fourth moment thus essentially boils down to bounding

$$\sum_f \left| \sum_n \tau(m)\psi_f(m) V(m) \right|^2$$
$$= \sum_f \sum_m \sum_n \tau(m)\tau(n)\psi_f(m)\psi_f(n) V(m,n),$$

for certain rapidly decaying functions V by the usual approximate functional equation argument.

DFI Theorem 1

We start with estimating the second moment of general linear forms

$$\mathcal{L}_f(\boldsymbol{a}) = \sum_{n \le N} a_n \psi_f(n).$$

We choose $\mathcal{F}$ to be an orthonormal basis of the space $S_k(\Gamma_0(q))$ with respect to the measure $y^{-2}\,dx\,dy$, $k \ge 2$.

Theorem (Theorem 1 of DFI)

For a sequence $\boldsymbol{a} = (a_n)$ *we have*

$$\sum_{f \in \mathcal{F}} |\mathcal{L}_f(\boldsymbol{a})|^2 = (k-1) \cdot [q + O(N \log N)] \cdot \|\boldsymbol{a}\|^2.$$

Notation: $\|\boldsymbol{a}\|^2 = \sum |a_n|^2$.

Proof of Theorem 1

We open up

$$S_q(\boldsymbol{a}) := \sum_{f \in \mathcal{F}} |L_f(\boldsymbol{a})|^2 = \sum_n \sum_m a_m \overline{a_n} \Delta(m, n),$$

where $\Delta(m, n) = \sum_{f \in \mathcal{F}} \psi_f(m) \overline{\psi_f(n)}$. Recall Petersson's formula

$$\begin{aligned}\Delta(m, n) =& (k-1) q \delta_{mn} \\ & + 2\pi i^k (k-1) q \sum_{q \mid c} \frac{S(n, m, c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right).\end{aligned}$$

To deal with Kloosterman sums we have the *Weil bound* and the *mean value theorem* for Dirichlet polynomials.

Bounding Kloosterman sums

Recall the mean value theorem

$$\sum_{d(\mathrm{mod} c)} \left| \sum_{n \leq N} b_n e\left(\frac{dn}{c}\right) \right|^2 = (c + O(N))\|\boldsymbol{b}\|^2.$$

From this we deduce using Cauchy-Schwarz that

$$\sum_{n \leq N} \sum_{m \leq N} b_m \overline{b_n} S(m, n; c) = \sum_{d(\mathrm{mod} c)} \sum_m b_m e(md/c) \sum_n \overline{b_n} e(n\overline{d}/c)$$
$$\ll (c + N)\|\boldsymbol{b}\|^2.$$

Using Weil's bound $S(m, n; c) \ll (m, n, c)^{1/2} c^{1/2} \tau(c)$ and the above we obtain

$$\sum_{n \leq N} \sum_{m \leq N} b_m \overline{b_n} S(m, n; c) \ll \min(c + N, c^{1/2} \tau^2(c) N)\|\boldsymbol{b}\|^2.$$

Finishing up

Let $q \gg N$. Using standard bounds of J_k we derive that

$$\sum_m \sum_n a_m \overline{a_n} S(m,n;c) J_{k-1}(4\pi\sqrt{mn}/c) \ll \frac{N}{c(k-1)!} \min(c, c^{1/2}\tau^2(c)N)\|a\|^2.$$

For finally summing over c note that

$$\sum_{q|c} c^{-2} \min(c, c^{1/2}\tau^2(c)N) \ll q^{-1} \log N,$$

so that

$$S_q(\boldsymbol{a}) = \left[q(k-1) + O\left(\frac{N \log N}{(k-2)!} \right) \right] \|\boldsymbol{a}\|^2.$$

This proves the theorem, at least for $q \gg N$. For the rest, see trick in DFI.

Corollary 1

We would like to bound more general sums

$$B(r,s) = \sum_m \sum_n a_m b_n \Delta(rm, sn) F(m,n),$$

where F is smooth, supported on $[M, 2M] \times [N, 2N]$ with $F^{(ij)} \ll M^{-i} N^{-j}$.

Corollary

If $(q, rs) = 1$ we have

$$B(r,s) \ll_{\varepsilon,k} (rsMN)^{\varepsilon} (q+M)^{1/2} (q+N)^{1/2} \|\boldsymbol{a}\| \|\boldsymbol{b}\|.$$

Important remark in the proof: From Petersson's formula we see that $\Delta(m,n)$ does not depend on the chosen ONB. So we choose an ONB of Hecke eigenfunctions.

Proof of Corollary 1

Extreme oversimplification: $\psi_f(m) = \lambda_f(m)$ and the λ_f's are completely multiplicative. Then

$$\sum_{m \asymp M} \sum_{n \asymp N} a_m b_m \Delta(rm, sn) = \sum \sum a_m b_m \sum_f \psi_f(rm)\psi_f(sn)$$
$$= \sum_f \psi_f(r)\psi_f(s) \sum_n a_n \psi_f(n) \sum_m b_m \psi_f(m)$$

We apply Deligne's bound $\psi_f(r) \ll r^{\varepsilon}$. For bounding F we use Fourier inversion: the analytic properties of F imply $\|\hat{F}\|_1 \ll 1$. Then we apply Theorem 1.

The (ugly) truth about Hecke eigenvalues

Computing with Euler factors or just the multiplicativity relations we derive

$$\lambda_f(an) = \sum_{\substack{a_1 n' = n \\ a_0 a_1 = a}} \mu(a_1)\lambda_f(a_0)\lambda_f(n').$$

Also

$$\tau(n)\lambda_f(n)\lambda_f(a) = \sum_{\substack{a_1 a_2^2 n' = n \\ a_0 a_1 a_2 = a}} \mu(a_1)\tau(a_1)\tau(n')\lambda_f(a_0 a_1 n')$$

and

$$\tau(m)\tau(n)\psi(rm)\psi(sn) = \sum_{\substack{\alpha\beta\gamma abc^2 = r \\ \gamma\delta = s, (\alpha\beta, \delta) = 1}} \mu(\beta)\mu(bc)\mu(c)\tau(\alpha)\tau(\beta)$$

$$\cdot \sum_{bc^2 d^2 m' = m} \sum_{\alpha\beta^2 n' = n} \tau(m')\tau(n')\psi(m')\psi(\beta\delta a n').$$

Theorem 2

Recall that for the subconvexity problem we need to bound the $B(r,s)$ in the special case

$$B(r,s) = \sum_m \sum_n \tau(m)\tau(n)\Delta(rm, sn)F(m,n).$$

We need a better bound than Corollary 1.

Theorem (Theorem 2 in DFI)

Let $(q, rs) = 1$ and $M, N \ll q^{1+\epsilon}$. We have

$$B(r,s) \ll q^{\epsilon}[q(r,s)(rs)^{-1/2} + q^{11/12}(rs)^{3/4}](MN)^{1/2}.$$

Proving Theorem 2 is hard. We start the proof today and Félicien will finish it and derive corollaries and subconvexity from it next time.

Petersson again and reduction to Kloosterman sums

We start with the case $r = 1$ and generalise later using the Hecke relations. We now have

$$B(s) = B(1, s) = \sum_{m} \sum_{n} \tau(m)\tau(n)\Delta(m, sn)F(m, n).$$

Using Petersson's formula we write

$$B(s) = (k-1)qT(0) + 2\pi i^k (k-1)q \sum_{c \equiv 0 (\mathrm{mod}\ q)} c^{-2} T(c),$$

where

$$T(0) = \sum_{n} \tau(sn)\tau(n)F(sn, n)$$

and

$$T(c) = c \sum_{m} \sum_{n} \tau(m)\tau(n)S(m, sn; c)J_{k-1}\left(\frac{4\pi\sqrt{smn}}{c}\right) F(m, n).$$

Estimating $T(0)$

Recall that $M, N \ll q$ and F has support on $[M, 2M] \times [N, 2N]$. Then

$$T(0) = \sum_n \tau(sn)\tau(n)F(sn, n) \ll \min(M/s, N)q^{\epsilon} \le \left(\frac{MN}{s}\right)^2 q^{\epsilon}.$$

This is good enough.

Splitting of $T(c)$

For $T(c)$ we execute the summation only over m first, using a Poisson/Voronoi type formula of Jutila.

Theorem (Jutila's summation formula)

Let g be a smooth compactly supported function on $\mathbb{R}^+$ and let $(c,d)=1$. Then we have

$$c\sum_{m=1}^{\infty}\tau(m)e\left(\frac{dm}{c}\right)g(m)=\int_0^{\infty}(\log x+2\gamma-2\log c)g(x)dx$$

$$-2\pi\sum_{m=1}^{\infty}\tau(m)e\left(\frac{-\bar{d}m}{c}\right)\int_0^{\infty}Y_0\left(\frac{4\pi\sqrt{mx}}{c}\right)g(x)$$

$$+4\sum_{m=1}^{\infty}\tau(m)e\left(\frac{\bar{d}m}{c}\right)\int_0^{\infty}K_0\left(\frac{4\pi\sqrt{mx}}{c}\right)g(x).$$

Splitting of $T(c)$

Accordingly, we split $T(c)$ as

$$T(c) = T^*(c) + T^-(c) + T^+(c),$$

where

$$T^*(c) = \sum_n S(0, sn; c) G^*(n)$$

and

$$\begin{aligned} T^\pm(c) &= \sum_m \sum_n \tau(m)\tau(n) S(0, sn \pm m; c) G^\pm(m, n) \\ &= \sum_h S(0, h; c) T_h^\pm(c), \end{aligned}$$

where $T_h^\pm(c) = \sum_{m \pm sn = h} \tau(m)\tau(n) G^\pm(m, n)$. Next time, we will talk about how to evaluate these sums.

What to remember for next time

We prove subconvexity in the level aspect using the fourth moment over a basis of Hecke eigenforms.
The fourth moment reduces to the second moment of linear forms with coefficients given by divisor functions (the second moment of convolution L-functions).
We also apply an amplifier, so we need to bound the more complicated $B(r, s)$.
This we have reduced to more tractable sums $T^{\pm}$, T^{*}, $T_h^{\pm}$, etc.
We bound this for $r = 1$ first, and then use Hecke relations to get the general result.

Bibliography

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