t-aspect subconvexity for GL_3 L-functions

Jesse Jääsaari

Queen Mary University of London

March 17, 2021

The main theorem

Our goal is to sketch the proof of the following result due to Munshi[1]:

Theorem

Let f be a Hecke-Maass cusp form for $\mathrm{SL}_3(\mathbb{Z})$. Then we have

$$L\left(\frac{1}{2}+it,f\right) \ll_{f,\varepsilon} (1+|t|)^{3/4-1/16+\varepsilon}.$$

Remarks:

• The convexity bound is $\ll_{f,\varepsilon} (1+|t|)^{3/4+\varepsilon}$.

• Subconvexity bound with the same exponent was known before by the work of Li in the case where f is self-dual (i.e. symmetric square lift of a GL_2 form) using the moment method.

• The above theorem is no longer state of the art result: Aggarwal[2] has shown an upper bound $\ll_{f,\varepsilon} (1+|t|)^{3/4-3/40+\varepsilon}$.

[1] JAMS, 2015

[2] Int. J. Number Theory, 2021

Maass cusp forms for $\mathrm{SL}_3(\mathbb{Z})$

- $\mathbb{H} \simeq \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$ so natural generalisation is $\mathbb{H}^3 \simeq \mathrm{SL}_3(\mathbb{R})/\mathrm{SO}_3(\mathbb{R})$.
- The group $\mathrm{SL}_3(\mathbb{R})$ acts on this space.

Definition

A smooth function $f : \mathbb{H}^3 \longrightarrow \mathbb{C}$ is a Maass cusp form for the group $\mathrm{SL}_3(\mathbb{Z})$ if

- $f(\gamma z) = f(z)$ for all $z \in \mathbb{H}^3$, $\gamma \in \mathrm{SL}_3(\mathbb{Z})$.
- f is an eigenfunction of every $\mathrm{SL}_3(\mathbb{Z})$-invariant differential operator on $\mathbb{H}^3$.
- f satisfies certain growth/cuspidality condition.

- Hecke theory also generalises. We say that f is a Hecke-Maass cusp form if it is also an eigenfunction for the Hecke algebra.
- As in the classical situation, GL_3 forms have Fourier expansions. The Fourier coefficients are indexed by pairs of integers and denoted by $\lambda(n_1, n_2)$.

• The L-series

$$L(s,f)=\sum_{m=1}^{\infty}\frac{\lambda(m,1)}{m^s}$$

converges for large enough $\Re(s)$ and has the usual nice properties (analytic continuation, functional equation, etc.).

• From the functional equation and the Phragmén-Lindelöf principle one gets the convexity bound

$$L\left(\frac{1}{2}+it,f\right)\ll_{f,\varepsilon}(1+|t|)^{3/4+\varepsilon}.$$

• The Ramanujan-Petersson conjecture is not known, but it holds on average (this follows from the Rankin-Selberg theory):

$$\sum_{m\leq x}|\lambda(m,1)|^2\ll_{\varepsilon}x^{1+\varepsilon}.$$

Kloosterman's δ-method

- Partition the circle using Farey fractions

$$\mathfrak{F}_Q = \left\{ \frac{a}{q} : 1 \leq q \leq Q,\, 1 \leq a < q,\, (a,q) = 1 \right\}.$$

- Farey fractions have several nice properties: two consecutive elements $a/q < a'/q'$ satisfy $a'/q' - a/q = 1/qq'$ and $q + q' > Q$. Two neighbouring mediants span an interval containing exactly one element of $\mathfrak{F}_Q$.
- This leads to Kloosterman's decomposition of the δ-symbol:

$$\delta(n) = 2\Re \int_0^1 \sum_{1 \leq q \leq Q < a \leq q+Q}\sum\nolimits^{\star} \frac{1}{aq} e\left(\frac{\overline{a}n}{q} - \frac{nx}{aq} \right) \mathrm{d}x.$$

- There are well-known drawbacks in this decomposition. For instance, the arithmetic and analytic parts are intertwined (unlike in DFI) and so it is (usually) difficult to treat the $a-$sum efficiently. This is not the case for us.
- The main advantage is the explicit form of the weight function $e(-nx/aq)$.

Reduction to weighted sum of Fourier coefficients

- Standard approximate functional equation argument shows that

$$L\left(\frac{1}{2}+it,f\right) \ll_\varepsilon t^\varepsilon \sup_{N\leq t^{3/2+\varepsilon}} \frac{|S(N)|}{N^{1/2}} + O\left(t^{-2021}\right),$$

where

$$S(N)=\sum_{n=1}^{\infty}\lambda(1,n)n^{-it}V\left(\frac{n}{N}\right)$$

for some smooth weight function V supported in $[1,2]$ and satisfying $V^{(j)}(x)\ll_j 1$.

- We will concentrate on the most difficult case $N \asymp t^{3/2}$.
- Denote

$$\delta(n)=\begin{cases}1 & \text{if } n=0\\ 0 & \text{otherwise}\end{cases} \qquad \text{and} \qquad \delta(|n|<X)=\begin{cases}1 & \text{if } |n|<X\\ 0 & \text{otherwise}\end{cases}$$

Conductor lowering mechanism

• The idea is to write $S(N)$ as a double sum

$$S(N) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \lambda(1, n) m^{-it} V\left(\frac{n}{N}\right) U\left(\frac{m}{N}\right) \delta(n-m),$$

where U is a smooth weight function having similar properties as V, and decompose the δ-symbol by using Kloosterman's version of circle method.

• This itself is not very effective in Munshi's treatment and so we need a conductor lowering mechanism.

• Kloosterman's δ-method picks the event $n = 0$ from the interval $[-N, N]$ by using $\asymp Q^2$ harmonics. As we have seen in previous talks, in practice we essentially need the number of oscillations to match the size of the equation, so in our case $Q^2 \asymp N$.

• Conductor lowering trick introduces more oscillations and hence reduces the size of the conductor.

• The idea is to add an extra factor $\delta(|n-m| < N/K)$ (which is redundant when $m = n$), for some parameter $t^{\varepsilon} \ll K \ll t$, to the double sum.
• One has

$$\delta(|n-m| < N/K) = \frac{1}{K} \int_{\mathbb{R}} \left(\frac{n}{m}\right)^{iv} W\left(\frac{v}{K}\right) \mathrm{d}v + O\left(t^{-2021}\right)$$

for $m, n \asymp N$ (here W is a smooth bump function).
• This follows from integration by parts.
• With this extra term the number of frequencies to detect $n = m$ in Kloosterman's δ-method is $\asymp Q^2 K$ so the optimal choice is $Q^2 \asymp N/K$, hence Q is smaller than initially.
• This is the most crucial "trick" in Munshi's paper.
• Nowadays it is understood that conductor lowering is actually built in Kloosterman's δ-method (cf. Aggarwal's paper).

- So write

$$
\begin{aligned}
S(N) &= \sum_{n=1}^{\infty} \lambda(1,n) n^{-it} V\left(\frac{n}{N}\right) \\
&= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \lambda(1,n) m^{-it} V\left(\frac{n}{N}\right) U\left(\frac{m}{N}\right) \delta(n-m) \delta(|n-m| < N/K) \\
&= S^{+}(N) + S^{-}(N),
\end{aligned}
$$

where

$$
S^{\pm}(N) = \frac{1}{K} \int_0^1 \int_{\mathbb{R}} W\left(\frac{v}{K}\right) \sum_{1 \le q \le Q < a \le q+Q} \sideset{}{^\star}\sum \frac{1}{aq} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \lambda(1,n) n^{iv} m^{-i(t+v)}
$$

$$
e\left(\pm \frac{(n-m)\overline{a}}{q} \mp \frac{(n-m)x}{aq} \right) V\left(\frac{n}{N}\right) U\left(\frac{m}{N}\right) \, \mathrm{d}v \, \mathrm{d}x.
$$

- The treatment of $S^{+}(N)$ and $S^{-}(N)$ is completely analogous.

- For convenience, set

$$W^\dagger(r,s) = \int\limits_0^\infty W(x)e(-rx)x^{s-1}\,\mathrm{d}x.$$

- For m-sum apply Poisson summation formula; conductor of the sum is $\asymp qt$ and length is N so essentially

$$\sum_{m=1}^\infty m^{-i(t+v)} e\left(-\frac{m\bar{a}}{q} + \frac{mx}{aq}\right) U\left(\frac{m}{N}\right)$$
$$\leftrightsquigarrow N^{1-i(t+v)} \sum_{\substack{|m|\ll qt/N \\ m\equiv \overline{a}\,(q)}} U^\dagger\left(\frac{N(ma-x)}{aq}, 1-i(t+v)\right).$$

- For n-sum apply GL_3 Voronoi summation formula; conductor of the sum is $\asymp q^3K^3$ and length is N so essentially

$$\sum_{n=1}^{\infty} \lambda(1,n) n^{iv} e\left(\frac{n\overline{a}}{q}\right) e\left(-\frac{nx}{aq}\right) V\left(\frac{n}{N}\right)$$

$$\leftrightsquigarrow N^{\frac{1}{2}+iv} \sum_{n \ll q^3K^3/N} \frac{\lambda(n,1)}{n^{1/2}} \frac{S(\overline{m}, n; q)}{q^{1/2}}$$

$$\cdot \int_{-K}^{K} \left(\frac{nN}{q^3}\right)^{-i\tau} \gamma\left(-\frac{1}{2}+i\tau\right) V^{\dagger}\left(\frac{Nx}{aq}, \frac{1}{2} - i(\tau - v)\right) \cdot \text{smooth fn}\, \mathrm{d}\tau,$$

where $\gamma(-1/2+i\tau)$ is a ration of Γ-factors.

- So in total (after executing the a-sum) $S(N) \leftrightsquigarrow N^{3/2-it} \sum_{q \asymp Q} \sum_{m \ll qt/N}$

$$\sum_{|n| \ll q^3K^3/N} \frac{\lambda(n,1)}{n^{1/2}} \cdot \frac{S(\overline{m}, n; q)}{aq^{3/2}} \int_{\mathbb{R}} W(v) \int_0^1 U^{\dagger}\left(\frac{N(ma-x)}{aq}, 1 - i(t+Kv)\right)$$

$$\int_{-K}^{K} \left(\frac{nN}{q^3}\right)^{-i\tau} \gamma\left(-\frac{1}{2}+i\tau\right) V^{\dagger}\left(\frac{Nx}{aq}, \frac{1}{2} - i\tau + iKv\right) \cdot \text{smooth fn}\, \mathrm{d}\tau\, \mathrm{d}x\, \mathrm{d}v.$$

• In the last display (and in what follows) a denotes the unique inverse of m modulo q in the interval $]Q, q+Q]$.
• We are now lead to analyse various integral transforms.
• By the stationary phase method (here the explicit form of the weight function is useful) we have

$$U^{\dagger}\left(\frac{N(ma-x)}{aq}, 1-i(t+Kv)\right) \asymp \frac{(t+Kv)^{1/2}aq}{N(x-ma)} \cdot \left(\frac{N(ma-x)}{aq(t+Kv)}\right)^{i(t+Kv)} \cdot \text{smooth fn}$$

and

$$V^{\dagger}\left(\frac{Nx}{aq}, \frac{1}{2}-i\tau+iKv\right) \asymp \left(\frac{aq}{Nx}\right)^{1/2} \cdot \left(\frac{Nx}{aq(Kv-\tau)}\right)^{i(\tau-Kv)} \cdot \text{smooth fn.}$$

• The v-integral can now be evaluated by a stationary phase analysis; it is essentially

$$\frac{1}{(t+\tau)^{1/2}K}\left(-\frac{(t+\tau)q}{Nm}\right)^{3/2-i(t+\tau)} \underbrace{\text{smooth fn}}_{\text{involves the x-integral}}$$

- We have now shown that

$$S(N)$$

$$\ll N^{3/2} \sum_{n\ll Q^3K^3/N} \frac{\lambda(n,1)}{n^{1/2}} \sum_{q\asymp Q} \sum_{|m|\ll qt/N} \frac{S(\overline{m},n;q)}{aq^{3/2}} \int_{-K}^{K} g(q,m,\tau) n^{-i\tau} \mathrm{d}\tau,$$

where $g(q,m,\tau)$ is essentially

$$\frac{1}{(t+\tau)^{1/2}K}\left(-\frac{(t+\tau)q}{Nm}\right)^{3/2-i(t+\tau)} \left(\frac{N}{q^3}\right)^{-i\tau} \gamma\left(-\frac{1}{2}+i\tau\right) \cdot \text{smooth fn.}$$

- Estimating trivially at this point gives $S(N) \ll K^{3/2}t^{1/2}Q^{3/2} \asymp N^{3/4}K^{3/4}t^{1/2} \asymp t^{13/8}K^{3/4}$ so actually K is hurting us here.

• The next step involves an application of Cauchy-Schwarz to get rid of the Fourier coefficients (by using that the Ramanujan-Petersson conjecture holds on average), but this process also squares the amount we need to save.

• At this point we roughly have

$$S(N) \ll N^{3/2} \left(\frac{Q^3 K^3}{N} \right)^{1/2}$$

$$\cdot \left(\sum_{n \asymp K^3 Q^3 / N} \frac{1}{n} \left| \sum_{q \asymp Q} \sum_{|m| \ll qt/N} \frac{S(\overline{m}, n; q)}{a q^{3/2}} \int_{-K}^{K} g(q, m, \tau) n^{-i\tau} \, \mathrm{d}\tau \right|^2 \right)^{1/2} .$$

• The idea is to open the absolute square, move the n-sum inside and execute it by using the Poisson summation formula.

• So the sum to consider is

$$\sum_{n \asymp K^3 Q^3 / N} \frac{1}{n^{1+i(\tau_1 - \tau_2)}} S(\overline{m_1}, n; q_1) S(\overline{m_2}, n; q_2) H\left(\frac{nN}{K^3 Q^3} \right)$$

for some compactly supported smooth weight function H.

- Note that length of the sum is $K^3Q^3/N \asymp N^{1/2}K^{3/2}$ (as $Q \asymp (N/K)^{1/2}$) and the conductor is $Q^2K \asymp N$. So the dual length will be $\ll N^{1/2}/K^{3/2}$.
- Here we are getting help from the parameter K as it makes the dual sum shorter.
- By Poisson summation formula the n-sum is

$$\frac{1}{q_1q_2}\left(N^{1/2}K^{3/2}\right)^{-i(\tau_1-\tau_2)}\sum_{n\ll N^{1/2}/K^{3/2}}\mathfrak{C}\cdot H^{\dagger}\left(\frac{nN^{1/2}K^{3/2}}{q_1q_2},-i(\tau_1-\tau_2)\right).$$

- Thus we are reduced to bounding

$$\sum_{q_1\asymp Q}\sum_{|m_1|\ll qt/N}\sum_{q_2\asymp Q}\sum_{|m_2|\ll qt/N}\sum_{n\ll N^{1/2}/K^{3/2}}|\mathfrak{C}|\cdot|\mathfrak{K}|,$$

where

$$\mathfrak{C}=\sum_{\beta\,(q_1q_2)}S(\overline{m_1},\beta;q_1)S(\overline{m_2},\beta;q_2)e\left(\frac{\beta n}{q_1q_2}\right)$$

and

$$\mathfrak{K} = \int_{\mathbb{R}} \int_{\mathbb{R}} \gamma\left(-\frac{1}{2}+i\tau_1\right)\overline{\gamma\left(-\frac{1}{2}+i\tau_2\right)} \frac{(N\cdot N^{1/2}K^{3/2})^{-i(\tau_1-\tau_2)}}{q_1^{-3i\tau_1}q_2^{3i\tau_2}} \cdot I(q_1,m_1,\tau_1)\overline{I(q_2,m_2,\tau_2)} H^{\dagger}\left(\frac{nN^{1/2}K^{3/2}}{q_1q_2}, -i(\tau_1-\tau_2)\right) \mathrm{d}\tau_1 \mathrm{d}\tau_2$$

with

$$I(q,m,\tau) = \frac{1}{(t+\tau)^{1/2}K}\left(-\frac{(t+\tau)}{Nm}\right)^{3/2-i(t+\tau)} V\left(-\frac{(t+\tau)q}{Nm}\right) \int_0^1 V\left(\frac{\tau}{K}-\frac{(t+\tau)x}{kma}\right)\mathrm{d}x.$$

- We have $\mathfrak{C} \ll q_1q_2(q_1,q_2,n)$. Furthermore, if $n=0$ then $\mathfrak{C}=0$ unless $q_1=q_2$ in which case $\mathfrak{C} \ll q_1^2(q_1,m_1-m_2)$. All this is elementary.
- Standard stationary phase analysis shows that

$$\mathfrak{K} \ll_{\varepsilon} \frac{1}{K^{3/2}t} \cdot \frac{N^{1/2}t^{\varepsilon}}{(|n|N^{1/2}K^{3/2})^{1/2}} \ll \frac{N^{1/4}t^{\varepsilon}}{t|n|^{1/2}K^{9/4}}$$

when $n \neq 0$ and

$$\mathfrak{K} \ll_\varepsilon \frac{t^\varepsilon}{Kt}$$

when $n = 0$.

- Explicit calculation shows that the total $n = 0$-contribution to $S(N)$ is

$$\ll_\varepsilon K^{1/2} N^{1/2} t^{1/2+\varepsilon} \asymp K^{1/2} t^{5/4+\varepsilon},$$

which is satisfactory if $K \ll t^{1/2}$.

- Similarly, the total $n \neq 0$-contribution to $S(N)$ is

$$\ll_\varepsilon \frac{N^{3/4} t^{1/2+\varepsilon}}{K^{1/2}} \asymp \frac{t^{13/8+\varepsilon}}{K^{1/2}},$$

which is satisfactory if $K \gg t^{1/4}$.

- The optimal choice $K \asymp N^{1/4} \asymp t^{3/8}$ leads to

$$S(N) \ll_\varepsilon t^{1/2+\varepsilon} N^{5/8} \ll t^{3/2-1/16+\varepsilon},$$

giving the claimed result.

- For smaller N optimisation is little different.

Aggarwal's improvement

• Aggarwal uses the same method as Munshi, but is able to bypass the conductor lowering trick by treating the x-integral more efficiently.
• This simplifies many of the arguments and leads to a better exponent.
• The point is that, as $a \asymp Q$, $q \leq Q$, $Q = (N/K)^{1/2}$, the x-integral in Kloosterman's δ-method works as a conductor lowering device:

$$\int_0^1 e\left(\frac{(n-m)x}{aq}\right) \mathrm{d}x \approx \int_0^1 e\left(\frac{(n-m)x}{Q^2}\right) \mathrm{d}x \approx \delta\left(|n-m| < N/K\right)$$

when $m, n \asymp N$.

The End