A mean-value theorem for Dirichlet polynomials via the amplification method

Dimitrios Lekkas

Department of Mathematics
University College London

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Definitions for Dirichlet polynomials

Let λ_n be a sequence of complex numbers and χ a character modulo q, then we define a Dirichlet polynomial as the sum:

$$D(s, \lambda, \chi) = \sum_{1 \leq n \leq N} \lambda_n \chi(n) n^{-s} \ .$$

We say that λ_n is a Dirichlet convolution of two sequences a_n and b_n and write $\lambda = \alpha \star \beta$ if $D(s, \lambda, \chi) = D(s, \alpha, \chi) D(s, \beta, \chi)$.
Let us also denote

$$\|\lambda\| = \sum_{\substack{n \leq N \\ (n,q)=1}} |\lambda_n|^2 \ .$$

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Seen in previous talks

So far we saw:

- Bounds on Exponential sums $S_f(n) = \sum_n e(f(n))$ via Weyl differencing
- Bounds on Character sums $S_H(\chi) = \sum_{x=N+1}^{N+H} \chi(x)$ due to Burgess

Today we'll talk about averages of Dirichlet polynomials $D(s, \lambda, \chi)$, following the paper "A Mean-Value Theorem for Character Sums" by J. Friedlander and H. Iwaniec.

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The Mean-Value Theorem for Dirichlet polynomials

The classical mean value theorem for Dirichlet polynomials asserts:

Theorem

$$S(\lambda) := \frac{1}{\phi(q)} \sum_{\chi(q)} \Big| \sum_{n \leq N} \lambda_n \chi(n) \Big|^2 = (1 + O(q^{-1}N)) \|\lambda\|$$

This result is best possible when $N < q$. Suppose now that $N > q$ and consider convolutions $\lambda = \alpha \star \beta \star \gamma$, with $N = KLM$ and $\alpha = (\alpha_k)_{k \leq K}, \beta = (\beta_l)_{l \leq L}$ and $\gamma = (\gamma_m)_{m \leq M}$, with α_k, β_l arbitrary and $\gamma_m = 1$. Let

$$S^*(\lambda) := \frac{1}{\phi(q)} \sum_{\substack{\chi(q) \\ \chi \neq \chi_0}} \Big| \sum_{n \leq N} \lambda_n \chi(n) \Big|^2$$

Then Friedlander and Iwaniec proved:

Theorem 1 (Friedlander,Iwaniec)

$$S^*(\lambda) \ll \|\alpha\|\|\beta\|\|\gamma\|(1 + q^{-3/4}(K+L)^{1/4}(KL)^{5/4} + q^{-1}(KL)^{7/4})q^{\epsilon}$$

Remarks:

- By using the Mean-Value theorem above and restricting $M \ll q^{1/2}$, the term $q^{-1}(KL)^{7/4}$ can be dropped.
- From the Polya-Vinogradov Theorem, we have:

$$\sum_{m \leq M} \chi(m) \ll q^{1/2} \log q \ ,$$

so that

$$\sum_{m \leq M} \chi(m) \ll M^{1-\delta}$$

for $M > q^{1/2+\epsilon}$ and some $\delta = \delta(\epsilon) > 0$, and

$$L(s,\chi) \ll q^{1/4} \log q$$

for $\Re s \geq \frac{1}{2}$,

- while Burgess managed to obtain

$$\sum_{m \leq M} \chi(m) \ll M^{1-\delta}$$

for $M > q^{1/4+\epsilon}$ and improved Dirichlet series bound, showing

$$L(s,\chi) \ll q^{3/16+\epsilon} \ .$$

Theorem 1 yields the following corollaries:

Corollary 1

Let χ be a non-principal character *mod p*. For $M > p^{5/11+\epsilon}$ we have

$$\sum_{m \leq M} \chi(m) \ll M^{1-\delta}$$

where δ and the implied constant may depend on ϵ.

Corollary 2

With χ as above, we have

$$L(s,\chi) \ll p^{5/11+\epsilon} \text{ for } \Re s \geq \frac{1}{2}$$

with an implied constant depending on ϵ and s.

These results give better bounds than the Polya-Vinogradov Theorem.
Both Theorem 1 and the Corollaries can be quantitatively sharpened using advanced tools, such as bounds on Kloosterman sums.
The authors focused on proving the above in the special case that:

- q is prime
- $\gamma(m) = f(m)$, where f is a smooth real function on $(\frac{1}{2}M, M)$
- $f^{(j)} \ll M^{-j}$, $j = 0, 1, 2, \ldots$

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Sketch of the proof

We can assume that $K, L, M < q$, otherwise use the Mean-Value Theorem. Let

$$S' = \frac{1}{q-1}\Big|\sum_{k\le K}\alpha_k\Big|^2\Big|\sum_{l\le L}\beta_l\Big|^2\Big|\sum_{m\le M}f(m)\Big|^2$$

be the sum that gives the contributions from the principal character. Then by opening the squares we have

$$S = S^* + S' = \sum\cdots\sum_{k_1 l_1 m_1 \equiv k_2 l_2 m_2 (q)} \alpha_{k_1}\bar{\alpha}_{k_2}\beta_{l_1}\bar{\beta}_{l_2} f(m_1) f(m_2)\ .$$

Now split S as

$$S = \sum_{|r|<R} S_r\ ,$$

where S_r is the sum including terms corresponding to $k_1 l_1 m_1 - k_2 l_2 m_2 = qr$ and $R = KLMq^{-1}$.
For $r = 0$ we have the trivial estimate

$$S_0 \ll \|\alpha\|\|\beta\|\|\gamma\| q^{\epsilon} \ .$$

For $r \neq 0$, set $\delta = (k_1 l_1, k_2 l_2), n_1 = k_1 l_1 \delta^{-1}, n_2 = k_2 l_2 \delta^{-1}$ and $s = r\delta^{-1}$.
Hence, $(n_1, n_2) = 1$ and $n_1 m_1 - n_2 m_2 = qs$. Equivalently,

$$m_1 \equiv qs\bar{n}_1 \mod n_2 \ .$$

Given δ, n_1, n_2, s sum over m_1 and apply Poisson's summation formula to get

$$\sum_{m_1,m_2} f(m_1)f(m_2) = \sum_{m_1 \equiv qs\bar{n}_1 (n_2)} f(m)f\Big(\frac{mn_1 - qs}{n_2}\Big)$$
$$= \frac{1}{n_1 n_2} \sum_h e\Big(-hqs\frac{\bar{n}_1}{n_2}\Big) \int f\Big(\frac{x}{n_1}\Big) f\Big(\frac{x-qs}{n_2}\Big) e\Big(\frac{hx}{n_1 n_2}\Big) dx$$

For $h = 0$ we have the main contribution:

$$T_s = \frac{1}{n_1 n_2} \int f\Big(\frac{x}{n_1}\Big) f\Big(\frac{x-qs}{n_2}\Big) dx \, .$$

After summing T_s over $s \neq 0$ we get

$$\begin{aligned}\sum_{s\neq 0} T_s &= \frac{1}{n_1 n_2}\int f\left(\frac{x}{n_1}\right)\sum_{s\neq 0} f\left(\frac{x-qs}{n_2}\right)dx \\ &= \frac{1}{n_1 n_2}\int f\left(\frac{x}{n_1}\right)\left(\int f\left(\frac{x-qs}{n_2}\right)+O(1)\right)dx \\ &= \frac{1}{q}\left(\int f(x)dx\right)^2 + O\left(\frac{M}{n_2}\right)\end{aligned} \tag{1}$$

By symmetry the error term can be shown to be $O\left(\frac{M}{n_1}\right)$, hence at the end we can have $O\left(\frac{M}{n_1+n_2}\right)$.

We conclude that the contribution to the sum S from these terms is

$$\frac{1}{q}\sum_{\substack{k_1,l_1\\k_2,l_2}}\alpha_{k_1}\bar{\alpha}_{k_2}\beta_{l_1}\bar{\beta}_{l_2}\left(\int f(x)dx\right)^2+O\left(M\sum_{\substack{k_1,l_1\\k_2,l_2}}\frac{|\alpha_{k_1}\alpha_{k_2}\beta_{l_1}\beta_{l_2}|(k_1l_1,k_2l_2)}{k_1l_1+k_2l_2}\right)$$

$$=\frac{1}{q}\sum_{\substack{k_1,l_1\\k_2,l_2}}\alpha_{k_1}\bar{\alpha}_{k_2}\beta_{l_1}\bar{\beta}_{l_2}\left(\sum_m f(m)+O(1)\right)^2+O(M\|\alpha\|\beta\|q^{\epsilon})$$

$$=S'+O(\|\alpha\|\|\beta\|\|\gamma\|(KLq^{-1}+q^{\epsilon})) \tag{2}$$

Thus the terms with $h=0$ cancel the main term S' apart from admissible error terms.

For the terms $h \neq 0$, we truncate and integrate by parts j times:

$$\begin{aligned} \int f\Big(\frac{x}{n_1}\Big) f\Big(\frac{x-qs}{n_2}\Big) e\Big(\frac{hx}{n_1 n_2}\Big) dx &\ll \Big(\frac{n_1 n_2}{h}\Big)^j \int \Big|\frac{d^j(ff)}{dx^j}\Big| dx \\ &\ll \Big(\frac{n_1 n_2}{h}\Big)^j \Big(\frac{1}{n_1} + \frac{1}{n_2}\Big)^j M^{-j} M \qquad (3) \\ &\ll \Big(\frac{n_1 + n_2}{hM}\Big)^j M \ll \Big(\frac{KL}{\delta hM}\Big)^j M \end{aligned}$$

by Leibniz' Rule and the bounds on $f^{(j)}$. Hence, if $|h| > H \doteq \left(\frac{KL}{M\delta}\right) q^{\epsilon}$ then the integral is $\ll (\delta hq)^{-2}$ and these terms contribute a negligible amount to S.

The remaining terms satisfy $O < |h| < H$ and contribute to S the sum $V = \sum_{0<\delta<R} V_\delta$, where

$$V_\delta = \sum_{\substack{O<|h|<H \\ 0<s<\frac{R}{\delta}}} \sum_{(k_1 l_1, k_2 l_2)=\delta} \alpha_{k_1} \bar{\alpha}_{k_2} \beta_{l_1} \bar{\beta}_{l_2} e\Big(-hqs \frac{\overline{k_1} \overline{l_1}/\overline{\delta}}{k_2 l_2/\delta} \Big) \\ \times \int f\Big(\frac{x\delta}{k_1 l_1}\Big) f\Big(\frac{x\delta - q\delta s}{k_2 l_2}\Big) e\Big(\frac{hx\delta^2}{k_1 l_1 k_2 l_2}\Big) \frac{\delta^2}{k_1 l_1 k_2 l_2} dx \;. \tag{4}$$

By using the properties of f one can use

$$\int |\hat{f}(y)| dy \ll \int_{-\infty}^{\infty} \min\Big(M, \frac{1}{y^2 M}\Big) dy \ll 1$$

and Cauchy-Schwarz to show

$$V_\delta \ll \frac{\delta_\tau(\delta) M}{KL} \Big(\frac{RH}{\delta}\Big)^{1/2} \|\alpha\| \|\beta\| \left(\frac{KL}{\delta^3} RHK(KL + RHK)(KL + L^2) \right)^{1/4} q^\epsilon \;.$$

Substituting for R and H and summing over δ we get

$$V \ll \|\alpha\|\|\beta\|\|\gamma\|(KL)^{5/4}q^{-1+\epsilon}(q+K^2L)^{1/4}(K+L)^{1/4}\ .$$

By the symmetry of the problem one can replace K^2L by $\min(K^2L, KL^2)$ and this is bounded by $\frac{(KL)^2}{K+L}$.This gives Theorem 1.

Proof of Corollary 1

To prove Corollary 1 take $q = p$ and $\alpha_k = \bar{\chi}(k), \beta_l = \bar{\chi}(l)(k \leq K, l \leq L)$. The contribution of $\chi \mod p$ to S^* is bounded below by

$$\frac{(KL)^2}{p}\left|\sum_{m \leq M} \chi(m)\right|^2 .$$

The contribution to S^* form each other non-principal character $\psi \mod p$ is greater than or equal to 0.
By choosing $K = L = p^{3/11}$ and using Theorem 1, we have

$$\left|\sum_{m \leq M} \chi(m)\right| \ll M^{1/2} p^{5/22+\epsilon} .$$