A factorization method for support characterization of an obstacle with a generalized impedance boundary condition

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Inverse problems: modeling and simulation,
Antalya, May 2012
The Generalized Impedance Boundary Conditions in acoustic scattering

\[ \frac{\partial u}{\partial \nu} + Zu = 0 \]

\[ \Delta u + k^2 u = 0 \]

\[ u = u^s + u^i \]

\[ \lim_{R \to \infty} \int_{|x|=R} \left| \frac{\partial u^s}{\partial r} - iku^s \right|^2 ds = 0 \]

Context:
- Imperfectly conducting obstacles
- Periodic coatings (homogenized model)
- Thin layers
- Thin periodic coatings
- ...
The Generalized Impedance Boundary Conditions in acoustic scattering

Context:
- Imperfectly conducting obstacles
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Inverse problem: recover $D$ from the scattered field.
General notions in inverse scattering

\[
\begin{aligned}
\Delta u^s + k^2 u^s &= 0 \\
\frac{\partial u^s}{\partial \nu} + Zu &= - \left( \frac{\partial u^i}{\partial \nu} + Zu^i \right) \text{ on } \Gamma \\
\lim_{R \to \infty} \int_{|x|=R} \left| \frac{\partial u^s}{\partial r} - iku^s \right|^2 ds &= 0
\end{aligned}
\]

\[
u^s(x) = \frac{e^{i kr}}{r^{(d-1)/2}} \left( u^\infty(\hat{x}) + O\left(\frac{1}{r}\right) \right) \quad r \to +\infty
\]
General notions in inverse scattering

\[
\begin{cases}
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\frac{\partial u^s}{\partial \nu} + Zu = -\left( \frac{\partial u^i}{\partial \nu} + Zu^i \right) \text{ on } \Gamma \\
\lim_{R \to \infty} \int_{|x|=R} \left| \frac{\partial u^s}{\partial r} - iku^s \right|^2 \, ds = 0
\end{cases}
\]

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u^s(x) = \frac{e^{ikr}}{r^{(d-1)/2}} \left( u^\infty(\hat{x}) + O\left( \frac{1}{r} \right) \right) \quad r \to +\infty
\]

For incident plane waves \( u^i(z, \hat{\theta}) = e^{ik\hat{\theta} \cdot z} \) we define

\[
u^\infty(\hat{x}, \hat{\theta}) \in \mathcal{L}^2(S^d, S^d).
\]
General notions in inverse scattering

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\begin{cases}
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For incident plane waves \(u^i(z, \hat{\theta}) = e^{ik\hat{\theta} \cdot z}\) we define

\(u^\infty(\hat{x}, \hat{\theta}) \in L^2(S^d, S^d)\).

Under minimal assumptions on \(Z\) design a method to recover \(D\) from \(u^\infty\) for all \((\hat{x}, \hat{\theta})\).
The factorization method: a sampling method

For \( u^i(x, \hat{\theta}) = e^{ik\hat{\theta} \cdot x} \) define

\[
(Z, D) \longrightarrow u^\infty(\hat{x}, \hat{\theta})
\]

where \( u^\infty \) associated with \( u^s(Z, D) \) is defined in dimension \( d \) by

\[
u^s(x) = \frac{e^{ikr}}{r^{(d-1)/2}} \left( u^\infty(\hat{x}) + O \left( \frac{1}{r} \right) \right) \quad r \rightarrow +\infty.
\]

Define the self-adjoint positive operator

\[
F : L^2(S^d) \longrightarrow L^2(S^d)
\]

\[
g \longmapsto \int_{S^d} u^\infty(\hat{x}, \hat{\theta}) g(\hat{\theta}) d\hat{\theta}
\]

\[
z \in D \iff e^{-ik\hat{\theta} \cdot z} \in \mathcal{R}(F_{\#}^{1/2})
\]

\[
F_{\#} := |\Re(F)| + \Im(F)
\]
The factorization method: a sampling method

For \( u^i(x, \hat{\theta}) = e^{ik\hat{\theta} \cdot x} \) define

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\[ F : L^2(S^d) \rightarrow L^2(S^d) \]

\[ g \mapsto \int_{S^d} u^\infty(\hat{x}, \hat{\theta}) g(\hat{\theta}) d\hat{\theta} \]

Define the self-adjoint positive operator

\[ F^\# := |\Re(F)| + \Im(F) \]

\[ z \in D \iff e^{-ik\hat{\theta} \cdot z} \in \mathcal{R}(F^{1/2}_\#) \]

\[ \exists g \text{ s.t. } F^{1/2}_\# g = e^{-ik\hat{\theta} \cdot z} \]
The factorization method: a sampling method

For $u^i(x, \hat{\theta}) = e^{ik\hat{\theta} \cdot x}$ define

$$(Z, D) \rightarrow u^\infty(\hat{x}, \hat{\theta})$$

where $u^\infty$ associated with $u^s(Z, D)$ is defined in dimension $d$ by

$$u^s(x) = \frac{e^{ikr}}{r^{(d-1)/2}} \left( u^\infty(\hat{x}) + O \left( \frac{1}{r} \right) \right) \quad r \rightarrow +\infty.$$

Define the self-adjoint positive operator

$$F : \quad L^2(S^d) \rightarrow L^2(S^d)$$

$$g \mapsto \int_{S^d} u^\infty(\hat{x}, \hat{\theta}) g(\hat{\theta}) \, d\hat{\theta}$$

No solution!
State of the art

- Factorization method for impenetrable scatterers:
  - Dirichlet and Neumann boundary condition: Kirsch 1998,
  - Impedance boundary condition ($Z = \lambda$): Kirsch & Grinberg 2002,
Outline

1. The GIBC forward problem
2. Characterization of scatterers via the factorization Theorem
3. Numerical examples
The GIBC forward problem

A volume formulation

• $V$ an Hilbert space such that $C^\infty(\Gamma) \subset V \subset H^{1/2}(\Gamma)$

• $Z : V \rightarrow V^*$ is linear and continuous and

$$Z^*u = \overline{Zu}$$

For example for complex functions $(\lambda, \mu) \in (L^\infty(\Gamma))^2$

$$Z = \text{div}_\Gamma \mu \nabla_\Gamma + \lambda$$

$$V = H^1(\Gamma)$$
The GIBC forward problem

A volume formulation

- \( V \) an Hilbert space such that \( C^\infty(\Gamma) \subset V \subset H^{1/2}(\Gamma) \)
- \( Z : V \rightarrow V^* \) is linear and continuous and \( Z^*u = \overline{Zu} \)
- \( \langle Zu, u \rangle_{V^*, V} \geq 0 \) for uniqueness reasons

The GIBC problem writes:

Find \( u^s \in \{ v \in D'(\Omega_{\text{ext}}), \varphi v \in H^1(\Omega_{\text{ext}}) \forall \varphi \in D(\mathbb{R}^d); v|_\Gamma \in V \} \)

\[
\begin{align*}
(P_{\text{vol}}) \quad \left\{ \begin{array}{l}
\Delta u^s + k^2 u^s = 0 \text{ in } \Omega_{\text{ext}}, \\
\frac{\partial u^s}{\partial \nu} + Zu^s = f \text{ on } \Gamma, \\
\lim_{R \to \infty} \int_{|x| = R} |\partial_r u^s - ik u^s|^2 = 0.
\end{array} \right.
\end{align*}
\]
The GIBC forward problem

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\end{cases}
\]

Find $u^s \in \{ v \in \mathcal{D}'(\Omega_{\text{ext}}), \varphi v \in H^1(\Omega_{\text{ext}}) \forall \varphi \in \mathcal{D}(\mathbb{R}^d); v|_{\Gamma} \in V \}$

The sign of the real part of the impedance operator is imposed by the volume equation!
Well posedness of the forward problem
A surface equivalent formulation

Find $u^s \in \{ v \in \mathcal{D}'(\Omega_{\text{ext}}), \varphi v \in H^1(\Omega_{\text{ext}}) \forall \varphi \in \mathcal{D}(\mathbb{R}^d); v|_{\Gamma} \in V \}$

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(P_{\text{vol}}) \begin{cases}
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\lim_{R \to \infty} \int_{|x|=R} |\partial_r u^s -iku^s|^2 = 0.
\end{cases}
\]

- $n_e : H^{1/2}(\Gamma) \rightarrow H^{-1/2}(\Gamma)$ the exterior DtN map
  \[ f \mapsto \frac{\partial u_f}{\partial \nu} \]
  where
  \[
  \begin{cases}
  \Delta u_f + k^2 u_f = 0 \text{ in } \Omega_{\text{ext}}, \\
u_f = f \text{ on } \Gamma, \\
\lim_{R \to \infty} \int_{|x|=R} |\partial_r u_f -iku_s|^2 = 0.
  \end{cases}
\]
Well posedness of the forward problem

A surface equivalent formulation

Find $u^s \in \{ v \in \mathcal{D}'(\Omega_{\text{ext}}), \varphi v \in H^1(\Omega_{\text{ext}}) \forall \varphi \in \mathcal{D}(\mathbb{R}^d); v_{|\Gamma} \in V \}$

\[ (P_{\text{vol}}) \quad \begin{cases} \Delta u^s + k^2 u^s = 0 \text{ in } \Omega_{\text{ext}}, \\ \frac{\partial u^s}{\partial \nu} + Zu^s = f \text{ on } \Gamma, \\ \lim_{R \to \infty} \int_{|x|=R} |\partial_r u^s -iku^s|^2 = 0. \end{cases} \]

- $n_e : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ the exterior DtN map
- $f \mapsto \frac{\partial u_f}{\partial \nu}$

\[ (P_{\text{vol}}) \iff (P_{\text{surf}}) \quad \begin{cases} \text{Find } u^s_{\Gamma} \in V \text{ such that} \\ (Z + n_e)u^s_{\Gamma} = f \end{cases} \]
Well posedness of the forward problem

A Fredholm operator

\[(\mathcal{P}_{\text{vol}}) \iff (\mathcal{P}_{\text{surf}})\]

\(\begin{aligned}
\{ & \text{Find } u_\Gamma^s \in V \text{ such that } \\
& (\mathbf{Z} + n_e)u_\Gamma^s = f \}
\end{aligned}\)

**Theorem**

If the embedding \(V \subset H^{1/2}(\Gamma)\) is compact and \(\mathbf{Z} = C_Z + K_Z\) with

- \(C_Z : V \to V^*\) isomorphism,
- \(K_Z : V \to V^*\) compact,

then \((\mathbf{Z} + n_e) : V \to V^*\) is an isomorphism.

**Proof**
Well posedness of the forward problem
A Fredholm operator

\[ (P_{\text{vol}}) \iff (P_{\text{surf}}) \quad \left\{ \begin{array}{l} \text{Find } u^s_\Gamma \in V \text{ such that} \\ (Z + n_e)u^s_\Gamma = f \end{array} \right. \]

**Theorem**

If the embedding \( V \subset H^{1/2}(\Gamma) \) is compact and \( Z = C_Z + K_Z \) with

- \( C_Z : V \to V^* \) isomorphism,
- \( K_Z : V \to V^* \) compact,

then \( (Z + n_e) : V \to V^* \) is an isomorphism.

**Proof**

- \( n_e : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma) \) is continuous,
Well posedness of the forward problem

A Fredholm operator

\[(P_{\text{vol}}) \iff (P_{\text{surf}})\]

\[
\begin{aligned}
\text{Find } u_{\Gamma}^s & \in V \text{ such that } \\
(Z + n_e)u_{\Gamma}^s & = f 
\end{aligned}
\]

**Theorem**

If the embedding \( V \subset H^{1/2}(\Gamma) \) is compact and \( Z = C_Z + K_Z \) with

- \( C_Z : V \to V^* \) isomorphism,
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then \( (Z + n_e) : V \to V^* \) is an isomorphism.

**Proof**

- \( n_e : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma) \) is continuous,
- hence \( Z + n_e : V \to V^* \) is Fredholm of index zero.
Well posedness of the forward problem

A Fredholm operator

\[(\mathcal{P}_{\text{vol}}) \iff (\mathcal{P}_{\text{surf}})\]
\[
\begin{cases}
\text{Find } u^s_\Gamma \in V \text{ such that } \\
(Z + n_e)u^s_\Gamma = f
\end{cases}
\]

**Theorem**

If the embedding \( V \subset H^{1/2}(\Gamma) \) is compact and \( Z = C_Z + K_Z \) with

- \( C_Z : V \to V^* \) isomorphism,
- \( K_Z : V \to V^* \) compact,

then \((Z + n_e) : V \to V^* \) is an isomorphism.

**Proof**

- \( n_e : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma) \) is continuous,
- hence \( Z + n_e : V \to V^* \) is Fredholm of index zero.
- Since \( \Im \langle Zu, u \rangle_{V, V^*} \geq 0 \), \((\mathcal{P}_{\text{vol}})\) is injective and so is \( Z + n_e \).
Outline

1. The GIBC forward problem
2. Characterization of scatterers via the factorization Theorem
3. Numerical examples
Implementation of the factorization method

1 First step: formal factorization

Find two bounded operators $G : \Lambda^* \to L^2(S^d)$ and $T : \Lambda \to \Lambda^*$ such that

$$F = GT^*G^*,$$

and

$$z \in D \iff \phi_z^\infty \in \mathcal{R}(G),$$

where $\phi_z^\infty(\hat{x}) := e^{-ikz \cdot \hat{x}}.$
Implementation of the factorization method

1. First step: formal factorization
Find two bounded operators $G : \Lambda^* \to L^2(S^d)$ and $T : \Lambda \to \Lambda^*$ such that

$$F = GT^*G^*,$$

and

$$z \in D \iff \phi_\infty^z \in \mathcal{R}(G),$$

where $\phi_\infty^z(\hat{x}) := e^{-ikz \cdot \hat{x}}$.

2. Second step: justification
Find the space $\Lambda$ and prove that

$$\mathcal{R}(G) = \mathcal{R}(F^{1/2}_\#).$$

$$F_\# = |\Re(F)| + \Im(F).$$
Formal factorization

Support characterization

Define the solving operator for the forward problem

\[ G : V^* \longrightarrow L^2(S^d) \]
\[ f \longmapsto u_f^\infty \]

where \( u_f^\infty \) is the far field associated with the solution to \((P_{\text{vol}})\) with \( f \) in the second hand side.

\[ z \in D \iff \phi_z^\infty \in \mathcal{R}(G) \]

**Hint:**

\[ G = G_{\text{Dir}} \circ (\mathbf{Z} + n_e)^{-1} \]

where \( G_{\text{Dir}} \) is the solving operator for the Dirichlet problem and

\[ z \in D \iff \phi_z^\infty \in \mathcal{R}(G_{\text{dir}}) \]
For $G_k(x) (= e^{ik|x|/|x|}$ in dimension 3) the radiating Green function for $\Delta + k^2$ define

$$\text{SL}_k(q)(x) = \int_{\Gamma} G_k(x - y)q(y)ds(y), \ x \in \mathbb{R}^d \setminus \Gamma,$$

$$\text{DL}_k(q)(x) = \int_{\Gamma} \frac{\partial G_k(x - y)}{\partial \nu(y)}q(y)ds(y), \ x \in \mathbb{R}^d \setminus \Gamma,$$

$$\left\{\begin{array}{l}
S_k := \text{SL}_k|\Gamma, \\
D_k := \text{DL}_k|\Gamma,
\end{array}\right.$$  
\quad and  
$$\left\{\begin{array}{l}
S'_k := \partial_\nu\text{SL}_k|\Gamma, \\
D'_k := \partial_\nu\text{DL}_k|\Gamma.
\end{array}\right.$$
Formal factorization
Definition of the central operator

For $G_k(x) (= e^{ik|x|/|x|}$ in dimension 3) the radiating Green function for $\Delta + k^2$ define

$$SL_k(q)(x) = \int_{\Gamma} G_k(x-y)q(y)ds(y), \ x \in \mathbb{R}^d \setminus \Gamma,$$

$$DL_k(q)(x) = \int_{\Gamma} \frac{\partial G_k(x-y)}{\partial \nu(y)}q(y)ds(y), \ x \in \mathbb{R}^d \setminus \Gamma,$$

$$\begin{cases} S_k := SL_k|_{\Gamma}, \\
D_k := DL_k|_{\Gamma}, \end{cases} \text{ and } \begin{cases} S'_k := \partial_\nu SL_k|_{\Gamma}, \\
D'_k := \partial_\nu DL_k|_{\Gamma}. \end{cases}$$

Thus we can deduce

$$T := ZS_k Z^* + D'_k + ZD_k + S'_k Z^*$$

$T$ has to be defined from $\Lambda$ to $\Lambda^*$ for some Hilbert space $\Lambda$. $\Lambda = V$ does not fit because $ZS_k Z^*$ not symmetric!
Formal factorization

*Difficulty in the definition of T*

\[ T := ZS_k Z^* + \mathcal{D}'_k + ZD_k + S'_k Z^* \]

\[ S_k : H^s(\Gamma) \rightarrow H^{s+1}(\Gamma) \]

We want \( T : \Lambda \rightarrow \Lambda^* \).

Consider

\[ Z = \Delta_\Gamma, \]

\[ V = H^1(\Gamma), \]

then by taking \( \Lambda = V \) we have:

\[ T : H^1(\Gamma) \rightarrow H^{-2}(\Gamma). \]

*Right space:* \( \Lambda = H^{3/2}(\Gamma) = \Delta^{-1}_\Gamma(H^{-1/2}(\Gamma)) \)
Careful definition of $T$ and rigorous factorization

If $V$ is compactly embedded into $H^{1/2}(\Gamma)$ define

$$\Lambda := \{ u \in V, Z^* u \in H^{-1/2}(\Gamma) \}$$

with

$$(u,v)_{\Lambda} := (u,v)_{H^{1/2}(\Gamma)} + (Z^* u, Z^* v)_{H^{-1/2}(\Gamma)}.$$

**Proposition**

- $Z + n_e : \Lambda \to H^{-1/2}(\Gamma)$ is an isomorphism,
- $G : \Lambda^* \to L^2(S^d)$ is continuous,
- $T : \Lambda \to \Lambda^*$ is continuous,

$$T = ZS_kZ^* + D'_k + ZD_k + S'_k Z^*$$

- $F = -GT^*G^*.$
Application of the factorization Theorem

Theorem [Grinberg 2002]

If \( F = -GT^*G^* \) with

1. \( G \) compact with dense range,
2. \( \Re(e^T) = C + K \) with \( C \) coercive and \( K \) compact,
3. \( -\Im(m(T^*)) \) compact and strictly positive on \( \mathcal{R}(G^*) \),

then \( \mathcal{R}(G) = \mathcal{R}(F^{1/2}) \).
Application of the factorization Theorem

Theorem [Grinberg 2002]

If \( F = -GT^*G^* \) with

1. \( G \) compact with dense range,
2. \( \Re(T) = C + K \) with \( C \) coercive and \( K \) compact,
3. \( -\Im(T^*) \) compact and strictly positive on \( \overline{\mathcal{R}(G^*)} \),

then \( \mathcal{R}(G) = \mathcal{R}(F^{1/2}_\#) \).

Conclusion: if \( k^2 \) is not an eigenvalue for the interior GIBC problem

\[ z \in D \iff \phi_z^\infty \in \mathcal{R}(F^{1/2}_\#) \]

\( V \) is compactly embedded into \( H^{1/2}(\Gamma) \),
\( Z \) is an admissible impedance boundary operator.
What if?

- Treated case: the embedding $V \subset H^{1/2}(\Gamma)$ is compact,

$$Z = \text{div}_\Gamma(\mu \nabla_\Gamma \cdot) + \lambda \cdot$$

$$V = H^1(\Gamma)$$

Symmetric case: the embedding $H^{1/2}(\Gamma) \subset V$ is compact,

$$Z = \lambda \cdot V = L^2(\Gamma)$$

Intermediate case: none of the compact embeddings hold.

$\Re(\mathcal{T})$ fails to be signed!
What if?

- Treated case: the embedding $V \subset H^{1/2}(\Gamma)$ is compact,
  \[ Z = \text{div}_\Gamma(\mu \nabla_\Gamma \cdot) + \lambda \cdot \]
  \[ V = H^1(\Gamma) \]

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What if?

- Treated case: the embedding $V \subset H^{1/2}(\Gamma)$ is compact,
  
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- Intermediate case: none of the compact embeddings hold.
  
  $\Re(T)$ fails to be signed!
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Numerical framework

- \( Z = \text{div}_\Gamma(\mu \nabla_\Gamma \cdot) + \lambda \cdot \),
- For \( N=50 \), the synthetic data are
  \[ \{ u_{i,j}^\infty \left( \frac{2i\pi}{N}, \frac{2j\pi}{N} \right) \} \quad i,j=1,\ldots,N \]
- The wavelength is equivalent to the size of the scatterer
- For each \( z \) in a given sampling grid we solve a discrete version of
  \[ F_\#^{1/2} g_z = \phi_z^\infty \]
  with Tikhonov-Morozov regularization and plot
  \[ z \mapsto \frac{1}{\|g_z\|}. \]
Influence of $\mu$

(a) $\mu = 100$

(b) $\mu = 1$

(c) $\mu = 0.1$
Influence of the wavelength

(d) $\mu = 1$, wavelength = 3

(e) $\mu = 1$, wavelength = 1.5

Thank you for your attention!
Influence of the wavelength

(d) $\mu = 1$, wavelength = 3

(e) $\mu = 1$, wavelength = 1.5

Thank you for your attention!