CONTAINMENT OF CERTAIN BRUHAT INTERVALS MODULO A MAXIMAL PARABOLIC SUBGROUP IN TYPE A

MATTHEW TOWERS

Let $W \cong S_n$ be the Weyl group of type A with generating set $\{s_1, \ldots, s_{n-1}\}$ where $s_i = (i, i+1)$, acting on the left on the set $\{1, 2, \ldots, n\}$. Permutations in W will be written in "complete form", that is we write

$$x = x_1 x_2 \cdots x_n$$

(or sometimes $x_1, x_2, \ldots, x_n$) where $x_i = x(i)$. Let $W_I \cong S_k \times S_{n-k}$ be the maximal parabolic subgroup generated by all s_i with $i \neq k$; both k and hence W_I are fixed throughout. Each coset xW_I contains a unique element $\bar{x}$ of minimal length with respect to these generators: $\bar{x}$ has the form $x_1 x_2 \cdots x_n$ where the sequences $(x_1, x_2, \ldots, x_k)$ and $(x_{k+1}, x_{k+2}, \ldots, x_n)$ are both increasing.

The object of this note is to prove the following result:

Theorem 1. *Let $x, q, w, y \in W$ be such that $q \leqslant \bar{x}$ and $y \leqslant \bar{w}$. Then*

$$[q, \bar{x}]W_I \subseteq [y, \bar{w}]W_I \tag{1}$$

if and only if there exists $z \in W_I$ such that $\bar{x}z \leqslant \bar{w}$ and $qz \geqslant y$.

Here $\leqslant$ denotes the Bruhat order, $[a, b]$ is the Bruhat interval $\{c \in W : a \leqslant c \leqslant b\}$ and $[a, b]W_I$ means $\{cW_I : c \in [a, b]\}$.

1. Definitions and lemmas

Before we begin on the proof we need a few definitions and lemmas on type A Bruhat order.

If $(a_1, a_2, \ldots)$ is some sequence of distinct natural numbers we write $\widetilde{(a_1, a_2, \ldots)}$ for the unique rearrangement of this sequence whose terms are increasing. On the set $\mathbb{N}^r$ we will use the product order $\leqslant'$ defined by $(a_1, a_2, \ldots) \leqslant' (b_1, b_2, \ldots)$ if and only if $a_i \leqslant b_i$ for all i. We say $(a_1, a_2, \ldots)$ **$\sim$-dominates** $(b_1, b_2, \ldots)$ if $\widetilde{(a_1, a_2, \ldots)} \geqslant' \widetilde{(b_1, b_2, \ldots)}$.

Lemma 2. *Let $w, y \in W$. For any $1 \leqslant m \leqslant n$ the following are equivalent:*

- *$w \geqslant y$ in the Bruhat order.*
- *$\widetilde{(w_1, \ldots, w_r)} \geqslant' \widetilde{(y_1, \ldots, y_r)}$ for all $r \leqslant m$ and $\widetilde{(w_r, \ldots, w_n)} \leqslant' \widetilde{(y_r, \ldots, y_n)}$ for all $r > m$.*

For $m = n$ this says $w \geqslant y$ if and only if $\widetilde{(w_1, \ldots, w_r)} \geqslant' \widetilde{(y_1, \ldots, y_r)}$ for all r.

Lemma 3. *Let $q \leqslant \bar{x}$. Then $q_i \leqslant \bar{x}_i$ for all $i \leqslant k$ and $q_i \geqslant \bar{x}_i$ for all $i > k$.*

Proof. For $i \leqslant k$ we have $\widetilde{(q_1, \ldots, q_i)} \leqslant' (x_1, \ldots, x_i)$ and the first part follows immediately. For the second, use Lemma 2 with $m = k$ and a similar argument. ☐

Lemma 4. *Suppose $a_1, \ldots, a_m$ $\sim$-dominates $b_1, \ldots, b_m$ and $a \geqslant b$. Then $a_1, \ldots, a_m, a$ $\sim$-dominates $b_1, \ldots, b_m, b$.*

Proof. Let $\widetilde{(a_1, \ldots, a_m)} = A_1, \ldots, A_m$ and $\widetilde{(b_1, \ldots, b_m)} = B_1, \ldots, B_m$, so that $A_i \geqslant B_i$ for all i. Let $A_r \leqslant a < A_{r+1}$ and $B_s \leqslant b < B_{s+1}$. We want to compare the sequences $A_1, \ldots, A_r, a, A_{r+1}, \ldots, A_m$ and $B_1, \ldots, B_s, b, B_{s+1}, \ldots, B_m$. Suppose first that $s \geqslant r$:

$$\begin{array}{ccccccccccccccc} \cdots & \leqslant & A_r & \leqslant & a & \leqslant & A_{r+1} & \leqslant & \cdots & \leqslant & A_{s-1} & \leqslant & A_s & \leqslant & A_{s+1} & \leqslant & \cdots \\ \cdots & \leqslant & B_r & \leqslant & B_{r+1} & \leqslant & B_{r+2} & \leqslant & \cdots & \leqslant & B_s & \leqslant & b & \leqslant & B_{s+1} & \leqslant & \cdots \end{array}$$

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Each thing in the bottom row is $\leqslant$ the thing directly above it. This is clear up to B_r and after B_{s+1}. In between, $B_{r+1} \leqslant b \leqslant a$, $B_{r+2} \leqslant b \leqslant a \leqslant A_{r+1}$, and so on up to $B_s \leqslant b \leqslant a \leqslant A_{s-1}$ and $b \leqslant a \leqslant A_s$.

Now suppose $s \leqslant r$

$$\begin{array}{ccccccccccccccccc} \cdots & \leqslant & A_s & \leqslant & A_{s+1} & \leqslant & A_{s+2} & \leqslant & \cdots & \leqslant & A_r & \leqslant & a & \leqslant & A_{r+1} & \leqslant & \cdots \\ \cdots & \leqslant & B_s & \leqslant & b & \leqslant & B_{s+1} & \leqslant & \cdots & \leqslant & B_{r-1} & \leqslant & B_r & \leqslant & B_{r+1} & \leqslant & \cdots \end{array}$$

Again we only need worry about the places between s and r. There, $b \leqslant B_{s+1} \leqslant A_{s+1}$, $B_{s+1} \leqslant A_{s+1} \leqslant A_{s+2}$ and so on up to $B_r \leqslant A_r \leqslant a$. ☐

An induction using the above Lemma shows that if $a_i \geqslant b_i$ for all $1 \leqslant i \leqslant m$ then for each $1 \leqslant i \leqslant m$, the sequence $a_1, \ldots, a_i$ $\sim$-dominates $b_1, \ldots, b_i$.

Corollary 5. *Suppose $u, v \in W$ and there exists $1 \leqslant m \leqslant n$ such that $u_i \leqslant v_i$ for all $1 \leqslant i \leqslant m$ and $u_i \geqslant v_i$ for all $i > m$ Then $u \leqslant v$.*

The following is taken from Fulton's *Young Tableaux* where it is Lemma 10.11 on p.174.

Lemma 6. *Let $u < v$ be permutations, let j be minimal such that $u_j \neq v_j$ (so $u_j < v_j$) and let m be minimal such that $m > j$ and $v_j > v_m \geqslant u_j$. Then $u \leqslant v(j,m) < v$.*

Here (j, m) is a transposition. Note that the complete form of $v(j,m)$ looks the same as that of v, except that the entries in positions j and m are swapped.

Lemma 7. *Let $q \leqslant \bar{x}$. Then for each $1 \leqslant r \leqslant k$ there is some permutation whose first k values are*

$$q_1, \ldots, q_r, x_{r+1}, \ldots, x_k$$

in the Bruhat interval $[q, \bar{x}]$.

Remark 8. *It's clear that Lemma 6 will help with the proof of this: e.g. suppose $q_1 < \bar{x}_1$, so $j = 1$ in the notation of Lemma 6. Look for m minimal such that $q_1 \leqslant \bar{x}_m < \bar{x}_1$, clearly we must have $m > k$. Now after position k, $\bar{x}$ looks like*

$$1, 2, 3, \ldots, \widehat{\bar{x}_1}, \ldots, \widehat{\bar{x}_2}, \ldots$$

(the hat denotes an omitted term). So in fact $\bar{x}_m = q_1$, and Lemma 6 gives

$$q \leqslant q_1 x_2 \cdots x_k | \cdots \leqslant \bar{x}$$

where what appears after x_k looks like the sequence $1, \ldots, n$ with $\bar{x}_1, \ldots, \bar{x}_k$ removed and then $\bar{x}_1$ substituted for q_1.

Proof. The proof is by induction on r, the base case being either vacuous (if $\bar{x}_1 = q_1$) or as discussed in the above remark. We need to strengthen the inductive hypothesis slightly: it will be that there is some element v in $[q, \bar{x}]$ of the form

$$q_1 \cdots q_r \bar{x}_{r+1} \cdots \bar{x}_k \cdots$$

where what appears after the kth place can be obtained by taking the sequence $1, \ldots, n$, deleting each of $\bar{x}_1, \ldots, \bar{x}_k$, then replacing some of $q_1, \ldots q_r$ with some of $\bar{x}_1, \ldots, \bar{x}_r$. Of course, some of $q_1, \ldots q_r$ may have been deleted as $\bar{x}_i$s. We do not assume q_i was replaced by $\bar{x}_i$.

If $\bar{x}_{r+1} = q_{r+1}$, the inductive step goes through immediately so we may as well assume $q_{r+1} < \bar{x}_{r+1}$. We apply Lemma 6 to $q < v$, its output will be between q and v so certainly in the interval $[q, \bar{x}]$. The first place in which q and v differ is $r + 1$, so this is the first element of the transposition occuring in Lemma 6. To find the second we must look for the first v_m in the interval $[q_{r+1}, \bar{x}_{r+1})$; clearly $m > k$.

Inductively the values of v from the kth place onwards look like

$$1, 2, 3, \ldots, \widehat{\bar{x}_1}, \ldots, \widehat{\bar{x}_2}, \ldots$$

with some of the $q_1, \ldots, q_r$ that remain replaced by some of $\bar{x}_1, \ldots, \bar{x}_r$. Thus the first v_m in $[q_{r+1}, \bar{x}_{r+1})$ is either q_{r+1} itself, or one of $\bar{x}_1, \ldots, \bar{x}_r$. If it was q_{r+1}, the inductive step goes through. Otherwise v_m is some $\bar{x}_*$ in the interval $(q_{r+1}, \bar{x}_{r+1})$. The result of applying Lemma 6 in this case is a permutation

$$q \leqslant q_1 \cdots q_r \bar{x}_* \bar{x}_{r+2} \cdots \bar{x}_k \cdots \leqslant \bar{x}$$

where the part of the permutation after place k is in the correct inductive form: we have swapped some $\bar{x}_*$ which was in the position of a q_* for $\bar{x}_{r+1}$.

Apply Lemma 6 repeatedly: each time we preserve the inductive form in places after k, each time we either put q_{r+1} in place $r+1$ or we put a *strictly smaller* $\bar{x}_*$ there. This can't go on forever, so eventually we get a permutation with q_{r+1} in place $r+1$, completing the inductive step. ☐

2. Only if

Suppose throughout this section that $q \leqslant \bar{x}, y \leqslant \bar{w}$, and that (1) holds. Thus

$$\forall u : q \leqslant u \leqslant \bar{x} \implies uW_I \cap [y, \bar{w}] \neq \emptyset. \tag{2}$$

Lemma 9. *The ith largest element of $y_1, \dots, y_k$ is dominated by at least i elements of $q_1, \dots, q_k$.*

Proof. Applying (2) with $u = q$ we see that $qz_1 \geqslant y$ for some $z_1 \in W_I$ Thus

$$(\widetilde{y_1, \dots, y_k}) \leqslant' (\widetilde{q_1, \dots, q_k})$$

and the result follows. ☐

Consider the following **proceedure P**. Initial step: choose $z(1)$ to be the minimal element of $\{1, \dots, k\}$ such that $q_{z(1)} \geqslant y_1$ if such an element exists, otherwise stop. General step: suppose the proceedure has constructed $z(1), \dots, z(m-1)$ successfully. Let $z(m)$ be the minimal element of $\{1, \dots, l\} \backslash \{z(1), \dots, z(m-1)\}$ such that $(\widetilde{q_{z(1)}, \dots, q_{z(m)}}) \geqslant' (\widetilde{y_1, \dots, y_m})$ if such an element exists, otherwise stop.

Lemma 10. *Proceedure P successfully constructs $z(1), \dots, z(k)$.*

Remark 11. *Applying (2) with $u = q$ we see that $qz_1 \geqslant y$ for some $z_1 \in W_I$, thus some q_* is greater than or equal to y_1 and $z(1)$ is defined. Let's look at the next step. What we need is the existence of some q_i with $z(1) \neq i \leqslant k$ such that $q_i, q_{z(1)}$ $\sim$-dominates y_1, y_2. If there is some q_* other than $q_{z(1)}$ that is $\geqslant y_2$, this will do by Lemma 4. If not, y_2 must be $\leqslant q_{z(1)}$ and is the largest of all $y_1, \dots, y_k$ by Lemma 9. Furthermore y_1 is at most the second largest, so it is dominated by a q_* other than $z(1)$, and this q_* together with $q_{z(1)}$ $\sim$-dominate y_1, y_2 by Lemma 3.*

Proof. As in the remark above, $z(1)$ is defined. Suppose that

- $z(1)$ is miminal such that $\widetilde{q_{z(1)} \geqslant y_1}$
- $z(2)$ is minimal such that $(\widetilde{q_{z(1)}, q_{z(2)}}) \geqslant (\widetilde{y_1, y_2})$
- ...
- $z(r)$ is mimimal such that $(\widetilde{q_{z(1)}, \dots, q_{z(r)}}) \geqslant (\widetilde{y_1, \dots, y_r})$

and $r < k$. We must show that the set of $q_* \in \{q_1, \dots, q_k\} \setminus \{q_{z(1)}, \dots, q_{z(r)}\}$ such that $q_{z(1)}, \dots, q_{z(r)}, q_*$ $\sim$-dominates $y_1, \dots, y_{r+1}$ is non-empty.

As before, if there is any element of $\{q_1, \dots, q_k\} \setminus \{q_{z(1)}, \dots, q_{z(k)}\}$ dominating y_{r+1} we are done. So we assume this fails, and therefore by Lemma 9 y_{r+1} is the rth largest (or larger) element of the set $y_1, \dots, y_k$. This mean that one of $y_1, \dots, y_r$ is only the $(r+1)$st largest (or smaller) of the set $y_1, \dots, y_k$, so is dominated by a q_* which is not any of the $q_{z(i)}$s. Take y_i to be the largest element of $y_1, \dots, y_k$ such that there exists $q_M \notin \{q_{z(1)}, \dots, q_{z(r)}\}$ with $M \leqslant k$ and $q_M \geqslant y_i$. We have $y_i < y_{r+1}$, otherwise $y_{r+1} < y_i < q_M$ contradicting our assumption.

Let $(\widetilde{y_1, \dots, y_r}) = Y_1, \dots, Y_r$ and $(\widetilde{q_{z(1)}, \dots, q_{z(r)}}) = Q_1, \dots Q_r$. Suppose $Y_{l-1} < y_{r+1} < Y_l$. We have the following diagram of inequalities:

$$\begin{array}{ccccccccccccc}
Q_1 & < & \cdots & < & Q_{l-1} & & & < & Q_l & < & \cdots & < & Q_r \\
\rotatebox{90}{$\leqslant$} & & & & \rotatebox{90}{$\leqslant$} & & & & \rotatebox{90}{$\leqslant$} & & & & \rotatebox{90}{$\leqslant$} \\
Y_1 & < & \cdots & < & Y_{l-1} & < & y_{r+1} & < & Y_l & < & \cdots & < & Y_r
\end{array}$$

y_i appears somewhere amongst $Y_1, \dots, Y_{l-1}$. Now y_{r+1} is at most the $(r-l+2)$th largest of $y_1, \dots, y_k$ so it is dominated by at least $(r-l+2)$ of $q_1, \dots, q_k$, all of which by assumption are Q_*s. It follows $y_{r+1} \leqslant Q_{l-1}$.

Say $y_i = Y_A$, where $A \leqslant l-1$. Each of $Y_{A+1}, \dots, Y_{l-1}$ is only dominated by elements of $q_1, \dots, q_k$ that are amongst our Q_* by definition of y_i. Furthermore if $Y_{l-1} \neq y_i$ then it is at most the $(r-l+3)$th largest of $y_1, \dots, y_k$, so it is dominated by at least $(r-l+3)$ of $q_1, \dots, q_k$ all of which are Q_*s, so it must be $\leqslant Q_{l-2}$. The same argument shows each Y_a is $\leqslant Q_{a-1}$ for $A+1 \leqslant a \leqslant l-1$. So:

$$\begin{array}{ccccccccccccccccccc}\cdots & < & Q_{A-1} & & q_M & & Q_A & < & \cdots & < & Q_{l-2} & < & Q_{l-1} & < & Q_l & < & \cdots & < & Q_r \\ & & \geqslant & & \geqslant & & \geqslant & & & & \geqslant & & \geqslant & & \geqslant & & & & \geqslant \\ \cdots & < & Y_{A-1} & < & y_i & < & Y_{A+1} & < & \cdots & < & Y_{l-1} & < & y_{r+1} & < & Y_l & < & \cdots & < & Y_r\end{array}$$

We have $(Q_1, \ldots, Q_r) \geqslant' (Y_1, \ldots, Y_{A-1}, Y_{A+1}, \ldots, Y_{l-1}, y_{r+1}, Y_l, \ldots, Y_r)$ and $q_M \geqslant y_i$. By Lemma 4, $(Q_1, \ldots, Q_r, q_M)$, a rearrangement of $(q_{z(1)}, \ldots, q_{z(r)}, q_M)$, $\sim$-dominates $(Y_1, \ldots, Y_r, y_{r+1})$ which is a rearrangement of $(y_1, \ldots, y_{r+1})$. This completes the proof. ☐

Note that $\{z(1), \ldots, z(k)\} = \{1, \ldots, k\}$, so we may think of z as a permutation of $\{1, \ldots, k\}$.

Lemma 12. *Let $1 \leqslant m \leqslant k$. No m-tuple from $q_1, \ldots, q_{z(m)-1}$ $\sim$-dominates $y_1, \ldots, y_m$.*

Remark 13. *In the case $q_{z(r)-1} < r$, this lemma says nothing. Let's see how it works for $r = 2$: suppose a pair $q_r < q_s$ from $q_1, \ldots, q_{z(2)-1}$ is such that $(q_r, q_s) \geqslant' (\widetilde{y_1, y_2})$. Neither r nor s can equal $z(1)$ otherwise we contradict the definition of $z(2)$. So we have $q_R \geqslant y_1, q_S \geqslant y_2$ for some $\{R, S\} = \{r, s\}$. We may replace q_R by $q_{z(1)}$ and preserve these inequalities, so by Lemma 4, $q_{z(1)}, q_S$ $\sim$-dominates y_1, y_2. This contradicts the definition of $z(2)$.*

Proof. Suppose some m-tuple $q_{a_1}, \ldots, q_{a_m}$ from $q_1, \ldots, q_{z(m)-1}$ $\sim$-dominates $y_1, \ldots, y_m$. We will show that some element of this m-tuple together with $q_{z(1)}, \ldots, q_{z(m-1)}$ form another m-tuple $\sim$-dominating $y_1, \ldots, y_m$, contradicting the minimality of $z(m)$.

Write $(\widetilde{y_1, \ldots, y_{m-1}}) = Y_1, \ldots, Y_{m-1}$, $(\widetilde{q_{z(1)}, \ldots, q_{z(m-1)}}) = Q_1, \ldots, Q_{m-1}$. Since $(\widetilde{y_1, \ldots, y_m}) \geqslant' (\widetilde{q_{a_1}, \ldots, q_{a_m}})$ there are $\{\alpha_1, \ldots, \alpha_m\} = \{a_1, \ldots, a_m\}$ such that

$$\begin{array}{ccc} Q_1 \geqslant & Y_1 & \leqslant q_{\alpha_1} \\ Q_2 \geqslant & Y_2 & \leqslant q_{\alpha_2} \\ \vdots & \vdots & \vdots \\ Q_{m-1} \geqslant & Y_{m-1} & \leqslant q_{\alpha_{m-1}} \\ & y_m & \leqslant q_{\alpha_m} \end{array}$$

If $q_{\alpha_m} \notin \{Q_1, \ldots, Q_{m-1}\} = \{q_{z(1)}, \ldots, q_{z(m-1)}\}$ then we have $\alpha_m < z(m)$ and $q_{z(1)}, \ldots, q_{z(m-1)}, q_{\alpha_m}$ $\sim$-dominates $y_1, \ldots, y_m$, a contradiction to minimality of $z(m)$. So we may assume $q_{\alpha_m} = Q_{M_0}$ some $1 \leqslant M_0 \leqslant m-1$.

Suppose $q_{\alpha_m} = Q_{M_0}, q_{\alpha_{M_0}} = Q_{M_1}, \ldots, q_{\alpha_{M_X}} = Q_{M_{X+1}}$, but $q_{\alpha_{M_{X+1}}} \notin \{Q_1, \ldots, Q_{m-1}\}$.

First, I claim that the $M_0, \ldots, M_{X+1}$ are pairwise distinct. Suppose this holds for $M_0, \ldots, M_L$ but $M_{L+1} = M_R$ where $R \leqslant L$. Then $q_{\alpha_{M_L}} = Q_{M_{L+1}} = Q_{M_R} = q_{\alpha_{M_{R-1}}}$ (or q_{α_m} if $R = 0$). But the q_* are pairwise distinct, so $M_{R-1} = M_L$ contradicting pairwise distinctness of $M_0, \ldots, M_L$.

We now have:

- $y_m \leqslant Q_{M_0}$
- $Y_i \leqslant Q_i$ if i is not one of the M_*
- $Y_{M_i} \leqslant q_{\alpha_{M_i}} = Q_{M_{i+1}}$ if $i \leqslant X$
- $Y_{M_{X+1}} \leqslant q_{\alpha_{M_{X+1}}} \notin \{Q_1, \ldots, Q_{m-1}\}$

It follows from Lemma 4 that $Y_1, \ldots, Y_{m-1}, y_m$ is $\sim$-dominated by $Q_1, \ldots, Q_{m-1}, q_{\alpha_{M_{X+1}}}$, hence by $q_{z(1)}, \ldots, q_{z(m-1)}, q_{\alpha_{M_{X+1}}}$. This contradicts the definition of $z(m)$. ☐

Lemma 14. *For each $1 \leqslant r \leqslant k$ we have $\bar{x}_{z(r)} \leqslant \bar{w}_r$.*

Proof. Lemma 7 combined with 2 show that there is some $u \in W$ whose first k values are

$$q_1 \cdots q_{z(r)-1} \bar{x}_{z(r)} \cdots \bar{x}_k$$

with the property that there exists $v \in uW_I$ such that $y \leqslant v \leqslant \bar{w}$. In particular,

$$(\widetilde{y_1, \ldots, y_r}) \leqslant' (\widetilde{v_1, \ldots, v_r}) \leqslant' (\bar{w}_1, \ldots, \bar{w}_r).$$

Not all of $(v_1, \ldots, v_r)$ can come from $q_1, \ldots, q_{z(r)-1}$ by Lemma 12. Thus one of $\bar{x}_{z(r)}, \ldots, \bar{x}_k$ appears amongst $v_1, \ldots, v_r$. In particular, the smallest such $\bar{x}_*$ namely $\bar{x}_{z(r)}$ is $\leqslant \bar{w}_r$. ☐

Corollary 15. *For each* $1 \leqslant r \leqslant k$ *we have* $(y_1, \widetilde{\ldots}, y_r) \leqslant' (qz(1), \widetilde{\ldots}, qz(r))$ *and* $(\bar{x}z(1), \widetilde{\ldots}, \bar{x}z(r)) \leqslant' (\bar{w}_1, \ldots, \bar{w}_r)$.

Proof. The statement about y_*s and $qz(*)$s is true by construction of z. The statement about $\bar{x}z(*)$s and $\bar{w}_*$s follows by Lemma 14 and an induction using Lemma 4. □

Similar arguments for the positions $k+1, \ldots, n$ will produce $z(k+1), \ldots, z(n)$ such that for all $k+1 \leqslant r \leqslant n$ we have $(y_r, \widetilde{\ldots}, y_n) \geqslant' (qz(r), \widetilde{\ldots}, qz(n))$ and $(\bar{x}z(r), \widetilde{\ldots}, \bar{x}z(n)) \geqslant' (\bar{w}_r, \ldots, \bar{w}_n)$. This completes the construction of the z required for our theorem by Lemma 2.

3. If

Suppose $q \leqslant \bar{x}, y \leqslant \bar{w}$ and that there exists $z \in W_I$ such that $\bar{x}z \leqslant \bar{w}$ and $y \leqslant qz$. We will give a proof that $[q, \bar{x}]W_I \subseteq [y, \bar{w}]W_I$ which works for any parabolic subgroup (not just maximal ones) and in any Coxeter group.

We can get a reduced expression for $\bar{x}z$ by concatenating reduced expressions for $\bar{x}$ and z. Since $q \leqslant \bar{x}$ this is a reduced expression for $\bar{x}z$ that contains a possibly non-reduced expression for qz, which can be refined to a reduced expression by omitting some terms. Thus $qz \leqslant \bar{x}z$.

Let $q \leqslant r \leqslant \bar{x}$, we need to show $rW_I \cap [y, \bar{w}]W_I \neq \emptyset$ and it is enough to show $rW_I \cap [qz, \bar{x}z]W_I \neq \emptyset$ because $y \leqslant qz \leqslant \bar{x}z \leqslant \bar{w}$. This we do by induction on the length of z, and the base case when $z = e$ is immediate.

Now let $z = z_0 s$ where s is a simple reflection in W_I and $l(z) = l(z_0) + 1$. By induction there is some $z' \in W_I$ such that $qz_0 \leqslant rz' \leqslant \bar{x}z_0$. We seek an element of $rW_I \cap [qz_0 s, \bar{x}z_0 s]$.

Case 1 $qz_0 s < qz_0$. Then

$$qz_0 s < qz_0 \leqslant rz' \leqslant \bar{x}z_0 < \bar{x}z_0 s.$$

Case 2.i $qz_0 s > qz_0, rz's > rz'$. Then

$$qz_0 s \leqslant rz's \leqslant \bar{x}z_0 s.$$

Case 2.ii $qz_0 s > qz_0, rz's < rz'$. Then $rz' > rz's, rz' \geqslant qz_0 < qz_0 s$, so by a lemma from Björner and Brenti, $qz_0 s \leqslant rz'$. Thus

$$qz_0 s \leqslant rz' \leqslant \bar{x}z_0 \leqslant \bar{x}z_0 s$$

This completes the proof.