Mathematical Vistas

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Maple Primary School is located near St Albans city center. Walking north from the commercial stretch of St Peters street, one passes a busy roundabout into the quiet segment in front of St Peters church, immediately after which a turn into Hall Place Gardens leads to the school grounds within two or three minutes. The school has a high standing among the locals, and has been viewed equally favorably by government inspectors. The excellent staff is led by Mr. Timothy Bowen, a head teacher of extraordinary accomplishment and commitment.

On Monday, 26 November, a nice presentation was delivered there by Ms. Jenny Gould, who works with the Hertfordshire education authorities. The topic was primary school mathematics, with the intention of helping parents to understand recent developments in pedagogical methodology.

The basic ideas that theorists of education in the UK regard as worthwhile seem roughly to run along the same lines that one finds in the United States. A good deal of effort goes towards making explicit very early on the actual reasoning behind the basic operations of arithmetic. A key theme, for example, is the decomposition

$$354 = 300 + 50 + 4$$

or better yet,

$$354 = \text{three hundreds, five tens, and four ones.}$$

In other words, teachers are expected to build into the curriculum an explicit understanding of place-value by way of arithmetic algorithms that take the pupils beyond the manipulation of syntax and into the meaning of the digits laid before them. Here is an example involving the addition of three largish numbers:

$$\begin{array}{r} 354 \\ +467 \\ +332 \\ \hline \\ 300 \\ 400 \\ 300 \\ \hline \\ 1200 \\ 50 \\ 60 \\ 30 \\ \hline \\ 140 \\ \hline \\ 1340 \\ 4 \\ 7 \\ 2 \\ \hline \\ 13 \\ \hline \\ 1353 \end{array}$$

What is termed the ‘compact method,’ that is, the method of adding digits and carrying, is taught only after a good level of semantic understanding is judged to have been attained through the absorption of this and other techniques.

It not easy for the general public to be fully aware of the discussion behind the scenes that determines educational agenda. In particular, it might surprise the concerned parent that the relative importance of manipulation and understanding can create debate at every level of learning, reaching right up to courses for Ph.D. students. In the U.S., much ink has been spilt in the last decade or so by pundits decrying the excesses of one pedagogical position or another, their ire stimulated by the ease of finding egregious examples of miseducation that can be blamed on an opposing camp. Utilizing the binary classification popular there, it is common now to characterize the 'conservatives' by their emphasis on repetition and drill, and some system of supposed 'basics' to which the children need to be herded back. Meanwhile, even at the risk of over-simplification, it might be reasonable to characterize the progressive wing of the discourse as advocates of *thinking*. To be against thinking may seem unthinkable, even as we consider the dead but distinguished opposition found in the personage of Alfred North Whitehead:

> Civilization advances by extending the number of important operations which we can perform without thinking about them. (Introduction to Mathematics, Chapter 5.)

Could this pithy case for automatic manipulation really be valid? Only in part, obviously. In the fashion typical for an English gentleman that delights in intellectual sparring, like his close associate Bertrand Russell, Whitehead overstates the case for the sake of emphasis, and thereby invites the reader to engage with him in instructive argumentation. For us, living day-to-day as teachers or parents, an immediate need for some kind of balance clearly outweighs the philosophical advantage of a dialectical struggle towards the truth.

The only component of Ms. Gould's presentation that might have been better done, therefore, also concerns what I perceived to be a hint of derision vis-a-vis the methodology of an older generation. While efforts to find the teaching ideas that encourage a suitable level of thinking is indeed laudable, it is all too easy to make satire of the automatic operative state, which, after all, is what most cognitive processes settle into as the user moves into progressively advanced stages of learning. Even with reservations about the martial imagery, one can appreciate Whitehead's point as he writes on:

> Operations of thought are like cavalry charges in a battle–they are strictly limited in number, they require fresh horses, and must only be made at decisive moments.

In considering the three different modes of the higher mental faculties,

(1) that of carrying out certain necessary operations,

(2) of understanding these operations at a deeper level,

and (3) of expressing this understanding,

ideologues and their theorists would do the public well to acknowledge plainly (and reasonably often) that they are not only complementary, but that they share indistinct boundaries. Although not immediately relevant for the challenges of day-to-day teaching, it might be useful also to call attention to creative minds in the mathematical establishment with notorious deficiencies in one or two of these modes, who are nevertheless valuable contributors to the community as a whole in the truest spirit of diversity.

Perhaps the endless debates themselves attest to the most uncontroversial fact about mathematics at this point in history, which is its overwhelming importance. Indeed the weight of the subject within human affairs grows at a rate that would have been hard to anticipate even two decades ago. The present state of affairs is owed in large measure to the on-going revolution in information theory (and practice) that presses the demand for technical literacy in everyday life ever more insistently, and threatens to leave the laggards forever behind. In the book cited above, Whitehead recalls Jonathan Swift's caricature of mathematicians as the rulers of Laputa. He goes on to point out that 'Swift might just as well have laughed at an earthquake,' contemporaneous as he was with Newton's Principia Mathematica and the momentous eruptions in quantitative technology that were to alter forever humankind's view of the universe.

What then is to become of our age very few would have the audacity to guess at with any seriousness. An important systemic difference from the Dawn of the Modern Age is that Newton was mostly preoccupied with *understanding* the world, whereas the engineers of our times are clearly more interested in *creating* new ones. The mainstream demand on conceptual machinery has undergone a corresponding shift that I have myself been astounded to observe in the classroom. After I entered the mathematical work-force, I ended up teaching post-graduate courses in abstract algebra at five-year intervals, and noticed the numbers of engineers enrolled increasing each time. They were profoundly interested in mathematical constructions that must have appeared utterly meaningless even to the most abstract physicist of the previous generation, such as translating a word or a picture into strings of one's and zeros like

$$00100100111010101011,$$

and then writing down a strange multiplication table for these strings, where one would have found answers like

$$111 \times 100 = 001.$$

Imagine a world of geometric objects, points, lines, circles, and polygons, whose dimensions are specified using these strange binary numbers, themselves representing small portions of a picture transmitted from Saturn by the Cassini probe! It is often remarked that Archimedes would have been entirely incredulous at the ability of most ordinary citizens of our times to perform arithmetic operations with minimal difficulty, a mental feat in his day as sophisticated as an understanding of $E = mc^2$ is to us. Thus, in spite of debates in education, or perhaps aided by them, knowledge is subject to the enormous escalations that rise to the demand of history and capital.

The Saturday following Ms. Gould's presentation, I had the privilege of attending open house at St Albans School, thereby, together with my teaching duties at University College, rounding out a basic introduction to the educational ladder of the United Kingdom. This secondary school likes to boast of its founding year, 948 AD, and the splendid grounds that are essentially contiguous with the beautiful abbey church of St Albans. The classics department is located inside the historic abbey gateway, from which one can look out through a double-arched window at a generous expanse of lawn that slopes down a hill towards Verulamium park. As one enters the auditorium, a modern building in contrast to much of its surroundings, prominent to the eye is a poster highlighting the career of a distinguished alumnus, the physicist Stephen Hawking.

The tour through the school as well as the greeting from the headmaster, Mr. Andrew Grant, painted a convincing picture of a humane institution with a deep tradition of high academic standards. Mr. Grant read English at Cambridge, concentrating on medieval literature, 17th century drama, and the Victorian poets, before his entry into the teaching profession. I had a pleasant encounter with him towards the end of the tour, in the course of which I learned that his son is studying maths at Cambridge. The adventurous young man is in fact in China for the moment teaching English, after which he hopes to return to complete his degree, and then settle into a stable profession. As might be expected, our conversation turned to the topic of mathematics and the teaching of it. I expressed then my wish to contribute to the school a presentation at some date agreeable to both sides, for the benefit or amusement of young people contemplating a scientific career.

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As my children progress through primary school, frequent inquiry is made by well-meaning acquaintances into my own approach to teaching in the home. That is to say, as increasing numbers of parents accept as inevitable a responsibility ostensibly (and arduously) shared with schools in the intellectual training of their children, a sharing of opinions and resources likewise pervades even casual contact with other parents, and much energy is poured into the incessant search for better methods. An awkward corollary is that even those habits of life governed largely by semi-conscious patterns, one is called upon to describe in some intelligible form, or to *justify.* When the obvious points of importance have already been made, that the children be mindful of life's sway, respectful of their teachers, and that they cultivate habits of study with a modicum of stability and focus, what more guidance is to be expected of a conscientious parent?

In hesitant attempts to render coherent my own inconsistent forays into the philosophy of teaching, I find myself imagining a kind of *grand and mysterious vista* in the foreground of further efforts. The vista, of course is the open landscape towards which we wave our hands, in the vague hope that worthwhile things will eventually be found in the blue mist, as is hinted at by the sunny meadow that leads into it. The mystery is the view on every horizon, there as soon as we ourselves are receptive to its presence, and in every nook and cranny that we stumble into with our children.

When a child begins to be comfortable with the multiplication table, one could delve into the $9\times$ entries, for instance, to find $9, 18, 27, 36, 45, 54, 63, 72, 81, 90$. Have your child add up the digits for each number, say, for 18, $1 + 8 = 9$. Eventually, the question of 'why?' should emerge spontaneously.

Many children I have met are surprisingly fond of the classification of numbers into even and odd. With just a little bit of patience, they can start summing up the odd numbers:

$$\begin{aligned}
&1 = 1\\
&1+3 = 4\\
&1+3+5 = 9\\
&1+3+5+7 = 16\\
&1+3+5+7+9 = 25\\
&\vdots
\end{aligned}$$

and even discern the pattern of the answers on the right. The *reason* for the pattern might be mulled over for quite a while and returned to several times before being properly understood.

The point is that at every stage, mathematics is fraught with *facts to be struck by*, plain to the eye that knows where to look. The examples here are taken from elementary arithmetic because it was the topic of Ms. Gould's presentation, and apparently the main worry of someone with a child in primary school. But in our own world and that of Plato, the interface with mystery evidently expands as the human sphere of knowledge itself grows out into the universe, and it is there that one finds intimations of grandeur.

My own father, who has thought about literature most of his life, once spoke to me about the essence of good poetry. Even after the definite details of the music, the imagery, and the philosophy have been understood or bypassed, a good poem, or those of lasting value that the best poets write a few times in their lives, must end up suggesting the *infinite*. The meaning and value of this notion may be far clearer to the lover of poetry than to the typical student of mathematics. After all, mathematicians often enough are carelessly proud of their ability to force even the different grades of infinity into a few symbols on paper. Precision and proof, the craftsmanship that goes into rendering intangibles in concrete form, are certainly indispensable tools of the trade.

So of course the children must add and multiply rather rapidly, and they must understand that in 1729, the second digit means seven hundred. It is probably also good if they can explain with suitable clarity and concision why the multiplication algorithm

$$\begin{array}{r}
234\\
314\\
\hline
936\\
234\\
702\\
\hline
73476
\end{array}$$

requires them to move each new line a step to the left.

But professional standards and zeal notwithstanding, it is unfortunate that even talented students at the university think of mathematics all too often as a collection of arid assertions to be proved upon demand. At least at the level of primary school, it is then perhaps incumbent upon parents to keep alive that part of educational development most concerned with the *soul* of learning, even as they rely on schools to tend mostly to the basic skills and a sound intellect. This then, is where the vista comes in, itself infinite in depth and extent, eager to offer up striking delights, sublime moods, and arduous pilgrimages in search of elusive goals. For my own children, I could well wish these things into their quotidian existence, intertwined with the chaotic strands of practical exigencies.

Alexander Grothendieck and Gerd Faltings are two influential mathematicians of our times. Faltings is known as a formidable architect of resolutions to the most difficult problems, while Grothendieck has very much the temperament of a prophet. Soon after the announcement of Faltings' most celebrated achievement, a proof of the so-called 'Mordell conjecture,' there was unexpected correspondence between the two minds. At this point already, Grothendieck was essentially in retirement, if not in hiding, and has since famously disappeared into Pyrenees. I reproduce here a portion of the letter.

> Dear Mr. Faltings,
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> Many thanks for your quick answer and for sending me your reprints! Your comments on the so-called 'Theory of Motives' are of the usual kind, and for a large part can be traced to a tradition that is deeply rooted in mathematics. Namely that research (possibly long and exacting) and attention is devoted only to mathematical situations and relations for which one entertains not merely the hope of coming to a provisional, possibly in part conjectural understanding of a hitherto mysterious region, as it has indeed been and should be the case in the natural sciences, but also at the same time the prospect of permanently supporting the newly gained insights by means of conclusive arguments. This attitude now appears to me as an extraordinarily strong psychological obstacle to the development of the visionary power in mathematics, and therefore also to the progress of mathematical insight in the usual sense, namely the insight which is sufficiently penetrating or comprehending to finally lead to a 'proof'. What my experience of mathematical work has taught me again and again, is that the proof always springs from the insight, and not the other way round, and that the insight itself has its source, first and foremost, in a delicate and obstinate feeling of the relevant entities and concepts and their mutual relations. The guiding thread is the inner coherence of the image which gradually emerges from the mist, as well as its consonance with what is known or foreshadowed from other sources, and it guides all the more surely as the exigence of coherence is stronger and more delicate. To return to Motives, there exists to my knowledge no 'theory' of motives, for the simple reason that nobody has taken the trouble to work out such a theory. There is an impressive wealth of available material both of known facts and anticipated connections, incomparably more, it seems to me, than ever presented itself for working out a physical theory! There exists at this time a kind of 'yoga des motifs', which is familiar to a handful of initiates, and in some situations provides a firm support for guessing precise relations, which can then sometimes be actually proved in one way or another (somewhat as, in your last work, the statement on the Galois action on the Tate module of abelian varieties). It has the status, it seems to me, of some sort of secret science, Deligne seems to me to be the person who is most fluent in it. His first [published] work, about the degeneration of the Leray spectral sequence for a smooth proper map between algebraic varieties over $\mathbf{C}$, sprang from a simple reflection on 'weights' of cohomology groups, which at that time was purely heuristic, but now (since the proof of the Weil conjectures) can be realized over an arbitrary base scheme. It is also clear to me that Deligne's generalization of Hodge theory finds for a large part its source in the unwritten 'Yoga' of motives, namely in the effort of establishing, in the framework of transcendent Hodge structures, certain 'facts' from this Yoga, in particular the existence of a filtration of the cohomology by 'weights', and also the semi-simplicity of certain actions of fundamental groups.

The language becomes progressively more abstruse in the subsequent paragraphs. My intention in quoting the letter therefore is not to comment on the mathematical ideas therein, close though they are to my current preoccupations. It is hoped rather that Grothendieck's candid expressions might convey even to the casual reader that curious sense of the unknown attached to any process of deep learning and thinking, and the urgent conviction at the core of worthwhile endeavor.

Meanwhile, I should admit that too much of an emphasis on the mystical dimension of learning can lead to confusion as easily as inspiration when it comes to classroom practice. Over the years, my own effectiveness as a teacher has occasionally suffered because of an innate predilection for the broad brush-strokes of philosophy versus the concrete details of workmanship. Outside the classroom, on the other hand, many pleasant hours of studying with my children have expanded effortlessly out of the search for vistas.