Diophantine geometry and non-abelian duality

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Leiden, May, 2011
Diophantine geometry and abelian duality

\[ E: \text{elliptic curve over a number field } F. \]

Kummer theory:

\[ E(F) \otimes \mathbb{Z}_p \longrightarrow H^1_f(G, T_p(E)) \]

conjectured to be an isomorphism.

Should allow us, in principle, to compute \( E(F) \).

Furthermore, size of \( H^1_f(G, T_p(E)) \) should be controlled by an \( L \)-function.
Diophantine geometry and abelian duality

In the theorem

$$L(E/\mathbb{Q}, 1) \neq 0 \Rightarrow |E(\mathbb{Q})| < \infty,$$

key point is that the image of

$$\text{loc}_p : H^1_f(G, T_p(E)) \rightarrow H^1_f(G_p, T_p(E))$$

is annihilated using Poitou-Tate duality by a class

$$c \in H^1(G, T_p(E))$$

whose image in

$$H^1(G_p, T_p(E))/H^1_f(G_p, T_p(E))$$

is non-torsion.
An explicit local reciprocity law then translates this into an analytic function on $E(\mathbb{Q}_p)$ that annihilates $E(\mathbb{Q})$.

$$\text{Exp}^* : H^1(G_p, T_p(E)) \sim F^1H^1_{DR}(E/\mathbb{Q}_p);$$

$$c \mapsto \frac{L(E, 1)}{\Omega(E)} \alpha$$

where $\alpha$ is an invariant differential form on $E$. 
Diophantine geometry and abelian duality

\[ \begin{array}{ccc}
E(\mathbb{Q}) & \rightarrow & E(\mathbb{Q}_p) \\
\downarrow & & \downarrow \\
H^1_f(G, T_p(E)) & \rightarrow & H^1_f(G_p, T_p(E)) \cup_c \mathbb{Q}_p
\end{array} \]
Non-abelian analogue?

Wish to investigate an extension of this phenomenon to *hyperbolic curves*. That is, curves of

- genus zero minus at least three points;
- genus one minus at least one point;
- genus at least two.
Notation

\(F\): Number field.

\(S_0\): finite set of primes of \(F\).

\(R \coloneqq \mathcal{O}_F[1/S_0]\), the ring of \(S\) integers in \(F\).

\(p\): odd prime not divisible by primes in \(S_0\); \(v\): a prime of \(F\) above \(p\) with \(F_v = \mathbb{Q}_p\).

\(G \coloneqq \text{Gal}(\bar{F}/F)\); \(G_v = \text{Gal}(\bar{F}_v/F_v)\).

\(G_S \coloneqq \text{Gal}(F_S/F)\), where \(F_S\) is the maximal extension of \(F\) unramified outside \(S = S_0 \cup \{v|p\}\).

\(\mathcal{X}\): smooth curve over \(\text{Spec}(R)\) with good compactification.
(Might be compact itself.)

\(\mathcal{X}\): generic fiber of \(\mathcal{X}\), assumed to be hyperbolic.

\(b \in \mathcal{X}(R)\), possibly tangential.
Suppose $\mathcal{X}$ is compact. Then the map

$$\hat{j} : X(R) \longrightarrow H^1(G, \hat{\pi}_1(\bar{X}, b));$$

$$x \mapsto [\hat{\pi}_1(\bar{X}; b, x)]$$

is a bijection.
Unipotent descent tower

\[
\begin{array}{c}
\ldots \\
\vdots \\
\ldots \\
\mathcal{X}(R) \\
\mathcal{X}(R)
\end{array}
\quad \quad \begin{array}{c}
\vdots \\
H^1_{(G, U_4)} \\
H^1_{(G, U_3)} \\
H^1_{(G, U_2)} \\
H^1_{(G, U_1)}
\end{array}
\]
Unipotent descent tower

\[ U = "\hat{\pi}_1(\bar{X}, b) \otimes \mathbb{Q}_p" \], is the \( \mathbb{Q}_p \)-pro-unipotent étale fundamental group of

\[ \bar{X} = X \times_{\text{Spec}(F)} \text{Spec}(\bar{F}) \]

with base-point \( b \).

The universal pro-unipotent pro-algebraic group over \( \mathbb{Q}_p \) equipped with a map from \( \hat{\pi}_1(\bar{X}, b) \).
Unipotent descent tower

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The map

\[ j : x \in \mathcal{X}(R) \mapsto [P(x)] \in H^1_f(G, U), \]

associates to a point \( x \), the \( U \)-torsor

\[ P(x) := \hat{\pi}_1(\bar{X}; b, x) \times \hat{\pi}_1(\bar{X}, b) U \]

of \( \mathbb{Q}_p \)-unipotent étale paths from \( b \) to \( x \).
Unipotent descent tower

\[ U_n := U^{n+1} \setminus U, \text{ where } U^n \text{ is the lower central series with } U^1 = U. \]

So \( U_1 = U^{ab} = T_p J_X \otimes \mathbb{Q}_p. \)
Unipotent descent tower

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So $U_1 = U^{ab} = T_p J_X \otimes \mathbb{Q}_p$.

All these objects have natural actions of $G$ so that $P(x)$ defines a class in

$$H^1_f(G, U),$$

the continuous non-abelian cohomology of $G$ with coefficients in $U$ satisfying local 'Selmer conditions', the *Selmer variety* of $X$, which must be controlled in order to control the points of $X$. 
Algebraic localization

\[ \mathcal{X}(R) \xrightarrow{j} \mathcal{X}(R_v) \]

\[ H^1_f(G, U_n) \xrightarrow{\text{loc}_v} H^1_f(G_v, U_n) \]
Algebraic localization

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\[ H^1_f(G, U_n) \quad \xrightarrow{\text{loc}_v} \quad H^1_f(G_v, U_n) \]

Goal:
Compute the image of \( \text{loc}_v \).
Algebraic localization

One essential fact is that the local map

\[ \mathcal{X}(R_v) \xrightarrow{j_V} H^1_f(G_v, U_n) \]

can be computed via a diagram

\[ \xymatrix{ \mathcal{X}(R_v) \ar[rr]^{j_V} \ar[dd]_{j_V} & & H^1_f(G_v, U_n) \ar[rr]^{\sim} & & U^{DR}_n / F^0 \ar[rr]^{\sim} & & \mathbb{A}^N } \]

where \( U^{DR}_n / F^0 \) is a homogeneous space for the De Rham-crystalline fundamental group, and the map \( j^{DR} \) can be described explicitly using \( p \)-adic iterated integrals.
Meanwhile, the localization map is an algebraic map of varieties over $\mathbb{Q}_p$ making it feasible, in principle, to discuss its computation.
Non-abelian method of Chabauty

Meanwhile, the localization map is an algebraic map of varieties over \( \mathbb{Q}_p \) making it feasible, in principle, to discuss its computation. Knowledge of

\[
\text{Im}(\text{loc}_v) \subset H^1_f(G_v, U_n)
\]

will lead to knowledge of

\[
\mathcal{X}(R) \subset [j_v]^{-1}(\text{Im}(\text{loc}_v)) \subset \mathcal{X}(R_v).
\]

For example, when \( \text{Im}(\text{loc}_v) \) is not Zariski dense, immediately deduce finiteness of \( \mathcal{X}(R) \).
Non-abelian method of Chabauty

This deduction is captured by the diagram

\[
\begin{array}{c}
\mathcal{X}(R) \xrightarrow{\text{jet}_v} \mathcal{X}(R_v) \\
\downarrow j^\text{et}_v \\
H^1_f(G, U_n) \xrightarrow{\text{loc}_v} H^1_f(G_v, U_n) \\
\downarrow \\
\exists \psi \neq 0 \\
\downarrow \\
\mathbb{Q}_p
\end{array}
\]

such that \( \psi \circ j^\text{et}_v \) kills \( \mathcal{X}(R) \).
Non-abelian method of Chabauty

Can use this to give a new proof of finiteness of points in some cases:

\( F = \mathbb{Q} \) and the Jacobian of \( X \) has potential CM. (joint with John Coates)

\( F = \mathbb{Q} \) and \( X \), elliptic curve minus one point.

\( F \) totally real and \( X \), of genus zero.
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\[ F \text{ totally real and } X \text{ of genus zero.} \]

In each of these cases, non-vanishing of a \( p \)-adic \( L \)-function seems to play a key role.
Non-abelian method of Chabauty

By analogy with the abelian case:

\[ \text{Non-vanishing of } L\text{-function} \Rightarrow \text{control of Selmer group} \]
\[ \Rightarrow \text{finiteness of points}; \]
Non-abelian method of Chabauty

By analogy with the abelian case:

*Non-vanishing of L-function $\Rightarrow$ control of Selmer group*$
\Rightarrow$ finiteness of points;

one has

*Non-vanishing of L-function $\Rightarrow$ control of Selmer variety*$
\Rightarrow$ finiteness of points.
Non-abelian method of Chabauty

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Non-vanishing of $L$-function $\Rightarrow$ control of Selmer group
$\Rightarrow$ finiteness of points;

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Non-vanishing of $L$-function $\Rightarrow$ control of Selmer variety
$\Rightarrow$ finiteness of points.

But would like to construct $\psi$ in some canonical fashion.
Motivation: Effective computation of points?

The goal is to find an effectively compute $m(X, v)$ such that

$$\min \{d_v(x, y) \mid x \neq y \in X(R) \subset X(R_v)\} > m(X, v).$$
Motivation: Effective computation of points?

The goal is to find an effectively compute \( m(X, \nu) \) such that

\[
\min\{d_\nu(x, y) \mid x \neq y \in \mathcal{X}(R) \subset \mathcal{X}(R_\nu)\} > m(X, \nu).
\]

Key point:

*Computation of \( m(X, \nu) \) + section conjecture \( \Rightarrow \) computation of \( \mathcal{X}(R) \).*
Motivation: Effective computation of points?

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$$\min\{d_v(x, y) \mid x \neq y \in \mathcal{X}(R) \subset \mathcal{X}(R_v)\} > m(X, v).$$

Key point:

*Computation of $m(X, v)$ + section conjecture $\Rightarrow$ computation of $\mathcal{X}(R)$.*

Remark: Section conjecture can also be used to effectively determine the existence of a point (A. Pal, M. Stoll).
Non-abelian duality?
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Hope that $Im(loc_v)$ might be computed using a sort of *non-abelian Poitou-Tate duality*. 
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Say $c \in H^1(G, V^*(1))$ has local component $c_w = 0$ for all $w \neq v$ and $c_v \neq 0$. Then $\text{Im}(H^1(G, V)) \subset H^1(G_v, V)$ lies in the hyperplane

\[(\cdot) \cup c_v = 0.\]

Would like a non-abelian analogue.
Non-abelian duality?

Hope that $\text{Im}(\text{loc}_v)$ might be computed using a sort of non-abelian \textit{Poitou-Tate duality}.

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Would like a non-abelian analogue.

Difficulty is that duality for Galois cohomology with coefficients in various non-abelian groups can be interpreted as a sort of \textit{non-abelian class field theory}. 
Non-abelian duality: example

$E/\mathbb{Q}$: elliptic curve with

$$\text{rank} E(\mathbb{Q}) = 1,$$

trivial Tamagawa numbers, and

$$|\text{III}(E)[p^{\infty}]| < \infty$$

for a prime $p$ of good reduction.

$X =: E \setminus \{0\}$ given as a minimal Weierstrass model:

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6.$$ 

So

$$X(\mathbb{Z}) \subset E(\mathbb{Z}) = E(\mathbb{Q}).$$
Non-abelian duality: example

Let

\[ \alpha = \frac{dx}{2y + a_1 x + a_3}, \quad \beta = \frac{x dx}{2y + a_1 x + a_3}. \]

Get analytic functions on \( X(\mathbb{Q}_p) \),

\[ \log_\alpha(z) = \int_b^z \alpha; \quad \log_\beta(z) = \int_b^z \beta; \]

\[ D_2(z) = \int_b^z \alpha \beta. \]

Here, \( b \) is a tangential base-point at 0, and the integral is (iterated) Coleman integration.
Locally, the integrals are just anti-derivatives of the forms, while for the iteration,

\[ dD_2 = (\int_b^z \beta) \alpha. \]
Theorem

Suppose there is a point \( y \in X(\mathbb{Z}) \) of infinite order in \( E(\mathbb{Q}) \). Then the subset

\[ X(\mathbb{Z}) \subset X(\mathbb{Z}_p) \]

lies in the zero set of the analytic function

\[
\psi(z) := D_2(z) - \frac{D_2(y)}{(\int_b^y \alpha)^2} (\int_b^z \alpha)^2.
\]
Non-abelian duality: example

**Theorem**

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*A fragment of non-abelian duality and explicit reciprocity.*
Non-abelian duality: example

Function $\psi$ is actually a composition

$$
\begin{array}{c}
\chi(\mathbb{Z}_p) \\ \xymatrix{
\ar[r] & H^1_f(G_p, U_2) \\ & \mathbb{Q}_p \\
& \mathbb{U}_2^{DR}/F^0}
\end{array}
$$

where $\phi$ is constructed using secondary cohomology products and has the property that

$$
\phi(\text{loc}_p(H^1_f(G, U_2))) = 0.
$$
Non-abelian duality: example

\[ \mathcal{X}(\mathbb{Z}) \longrightarrow H^1_f(G, U_2) \]

\[ \mathcal{X}(\mathbb{Z}_p) \longrightarrow H^1_f(G_p, U_2) \]

\[ U_2^{DR}/F^0 \]

\[ \sim \]

\[ \varphi \]

\[ \phi \]

\[ \mathbb{Q}_p \]
Non-abelian duality: example

\[ U_2 \cong V \times \mathbb{Q}_p(1) \]

where \( V = T_p(E) \otimes \mathbb{Q}_p \), with group law

\[(X, a)(Y, b) = (X + Y, a + b + (1/2) \langle X, Y \rangle) .\]

A function

\[ a = (a_1, a_2) : G_p \rightarrow U_2 \]

is a cocycle if and only if

\[ da_1 = 0; \quad da_2 = -(1/2)[a_1, a_1]. \]
Non-abelian duality: example

For \( a = (a_1, a_2) \in H_f^1(G_p, U_2) \), we define

\[
\phi(a_1, a_2) := [b, a_1] + \log \chi_p \cup (-2a_2) \in H^2(G_p, \mathbb{Q}_p(1)) \cong \mathbb{Q}_p,
\]
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where

\[
\log \chi_p : G_p \to \mathbb{Q}_p
\]

is the logarithm of the \( p \)-adic cyclotomic character and

\[
b : G \to V
\]

is a solution to the equation

\[
db = \log \chi_p \cup a_1.
\]
Non-abelian duality: example

The annihilation comes from the standard exact sequence

\[ 0 \rightarrow H^2(G, \mathbb{Q}_p(1)) \rightarrow \sum_v H^2(G_v, \mathbb{Q}_p(1)) \rightarrow \mathbb{Q}_p \rightarrow 0. \]

That is, our assumptions imply that the class

\[ [\pi_1(\bar{X}; b, x)]_2 \]

for \( x \in X(\mathbb{Z}) \) is trivial at all places \( l \neq p \).

On the other hand

\[ \phi(\text{loc}_p([\pi_1(\bar{X}; b, x)]_2)) = \text{loc}_p(\phi^{\text{glob}}([\pi_1(\bar{X}; b, x)]_2)). \]
Non-abelian duality: example

With respect to the coordinates

$$H^1_f(G_p, U_2) \simeq U^{DR}_2 / F^0 \simeq \mathbb{A}^2 = \{(s, t)\}$$

the image

$$\text{loc}_p(H^1_f(G, U_2)) \subset H^1_f(G_p, U_2)$$

is described by the equation

$$t - \frac{D_2(y)}{(\int_y \alpha)^2} s^2 = 0.$$
Let
\[ L = \bigoplus_{n \in \mathbb{N}} L_n \]
be graded Lie algebra over field \( k \). The map \( D : L \rightarrow L \) such that
\[ D|_{L_n} = n \]
is a derivation, i.e., an element of \( H^1(L, L) \). Can be viewed as an element of \( H^2(L^* \ltimes L, k) \), that is, a central extension of \( L^* \ltimes L \):
\[
0 \rightarrow k \rightarrow E' \rightarrow L^* \ltimes L \rightarrow 0.
\]
Non-abelian duality: abstract framework

Explicitly described as follows:

\[ [(a, \alpha, X), (b, \beta, Y)] = (\alpha(D(Y)) - \beta(D(X)), \text{ad}_X(\beta) - \text{ad}_Y(\alpha), [X, Y]). \]

When \( L = L_1 \) and \( D = I \), then this gives a standard Heisenberg extension.
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\[ [(a, \alpha, X), (b, \beta, Y)] = (\alpha(D(Y)) - \beta(D(X)), \text{ad}_X(\beta) - \text{ad}_Y(\alpha), [X, Y]). \]

When \( L = L_1 \) and \( D = I \), then this gives a standard Heisenberg extension.

When \( k = \mathbb{Q}_p \) and we are given an action of \( G \) or \( G_v \), can twist to

\[
0 \longrightarrow \mathbb{Q}_p(1) \longrightarrow E \longrightarrow L^*(1) \rtimes L \longrightarrow 0.
\]

Also have a corresponding group extension

\[
0 \longrightarrow \mathbb{Q}_p(1) \longrightarrow \mathcal{E} \longrightarrow L^*(1) \rtimes U \longrightarrow 0.
\]

\((L = \text{Lie}(U))\)
Non-abelian duality: abstract framework

From this, we get a boundary map

\[ H^1(G_v, L^*(1) \rtimes U) \xrightarrow{\delta} H^2(G_v, \mathbb{Q}_p(1)) \sim \mathbb{Q}_p. \]

This boundary map should form the basis of (unipotent) non-abelian duality.
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Non-abelian duality: difficulties

1. How to get functions on

\[ H^1_f(G_v, U) \]
Non-abelian duality: difficulties

1. How to get functions on

\[ H^1_f(G_v, U) \]?

2. When \( U \) is a unipotent fundamental group, \( L \) is not graded in way that's compatible with the Galois action.
Non-abelian duality: weighted completions

This second difficulty is partially resolved by Hain’s theory of weights completions.
Non-abelian duality: weighted completions

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Let \( R \) be the Zariski closure of the image of

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G_S \rightarrow \text{Aut}(H_1(\bar{X}, \mathbb{Q}_p)).
\]
Non-abelian duality: weighted completions

This second difficulty is partially resolved by Hain’s theory of weights completions.

Let $R$ be the Zariski closure of the image of

$$G_S \to \text{Aut}(H_1(\bar{X}, \mathbb{Q}_p)).$$

Then $R$ contains the center $\mathbb{G}_m$ of $\text{Aut}(H_1(\bar{X}, \mathbb{Q}_p))$. 
Non-abelian duality: weighted completions

Consider the universal pro-algebraic extension

\[ 0 \rightarrow T \rightarrow G_S \rightarrow R \rightarrow 0 \]

equipped with a lift

\[ G_S \rightarrow R \]

such that \( T \) is pro-unipotent and the action of \( \mathbb{G}_m \) on \( H_1(T) \) has negative weights.
Non-abelian duality: weighted completions

Then

$$H^1(G_S, U) \simeq H^1(G_S, U).$$

Easy to see by comparing the splittings of the two rows in

\[ \begin{array}{cccccc}
1 & \rightarrow & U & \rightarrow & U \rtimes G_S & \rightarrow & G_S & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & U & \rightarrow & U \rtimes G_S & \rightarrow & G_S & \rightarrow & 1
\end{array} \]
Furthermore, the exact sequence

$$0 \to T \to G \to S \to R \to 0$$

splits to give $\tilde{R} \subset G_S$ that maps isomorphically to $R$, and

$$G_S \cong T \ltimes \tilde{R}.$$ 

In particular, there is a lifted one-parameter subgroup $\mathbb{G}_m \subset G_S$, which gives a grading on all $G_S$ modules. (Actually, the $\mathbb{G}_m$-lifting determines $\tilde{R}$.)
Corollary

Let $N = \text{Lie} T$. The lifting $\mathbb{G}_m \subset G_S$ determines a grading on

$$[N^*(1) \times L^*(1)] \rtimes L \rtimes N.$$
Non-abelian duality: weighted completions

Corollary

Let $N = \text{Lie} T$. The lifting $\mathbb{G}_m \subset G_S$ determines a grading on

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One can use this to construct, in turn, central extensions of

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\[
[N^*(1) \times L^*(1)] \rtimes U \rtimes T;
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and

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[N^*(1) \times L^*(1)] \rtimes U \rtimes G_S;
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Non-abelian duality: weighted completions

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One can use this to construct, in turn, central extensions of

$$[N^*(1) \times L^*(1)] \rtimes L \ltimes N;$$

$$[N^*(1) \times L^*(1)] \rtimes U \ltimes T;$$

and

$$[N^*(1) \times L^*(1)] \rtimes U \ltimes G_S;$$

which can then be pulled back to

$$[N^*(1) \times L^*(1)] \rtimes U \ltimes G_S.$$
Non-abelian duality: weighted completions

Proposition

The $\mathbb{G}_m$ lift determines a central extension

$$0 \rightarrow \mathbb{Q}_p(1) \rightarrow \mathcal{E} \rightarrow [N^*(1) \times L^*(1)] \rtimes U \rtimes G_S \rightarrow 0$$

giving rise to a boundary map

$$H^1(G_S, [N^*(1) \times L^*(1)] \rtimes U) \rightarrow H^2(G_S, \mathbb{Q}_p(1)).$$
Non-abelian duality: weighted completions

The central extension can be pulled back to each $G_w$ for $w \in S$, to give boundary maps

$$H^1(G_w, N^*(1) \times L^*(1)) \rightarrow H^1(G_w, [N^*(1) \times L^*(1)] \rtimes U) \xrightarrow{\delta_w} \mathbb{Q}_p$$

$$H^1(G_w, U)$$
Non-abelian duality: remarks

1. Applying the constructions to the finite level quotients, we get maps

\[ H^1(G_w, [N_n^*(1) \times L_n^*(1)] \rtimes U_n) \xrightarrow{\delta_{w,n}} \mathbb{Q}_p \]

and

\[ H^1(G_S, [N_n^*(1) \times L_n^*(1)] \rtimes U_n) \xrightarrow{\delta_n} H^2(G_S, \mathbb{Q}_p(1)). \]
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and

\[ H^1(G_S, [N_n^*(1) \times L_n^*(1)] \rtimes U_n) \xrightarrow{\delta_n} H^2(G_S, \mathbb{Q}_p(1)). \]

2. These maps are compatible in the following sense: \( \delta_n \) restricted to

\[ H^1(G_S, [N_{n-1}^*(1) \times L_{n-1}^*(1)] \rtimes U_n) \]

is the composition

\[ H^1(G_S, [N_{n-1}^*(1) \times L_{n-1}^*(1)] \rtimes U_n) \xrightarrow{\delta_{n-1}} H^1(G_S, [N_{n-1}^*(1) \times L_{n-1}^*(1)] \rtimes U_{n-1}) \xrightarrow{\delta_{n-1}} H^2(G_S, \mathbb{Q}_p(1)); \]

and the same for the local versions.
Non-abelian duality: remarks

\[
H^1(G_w, [N^*_{n-1}(1) \times L^*_{n-1}(1)] \rtimes U_n) \leftrightarrow H^1(G_w, [N^*_n(1) \times L^*_n(1)] \rtimes U_n) \quad \downarrow \quad Q_p
\]

\[
H^1(G_w, [N^*_{n-1}(1) \times L^*_{n-1}(1)] \rtimes U_{n-1}) \rightarrow
\]

Non-abelian duality: remarks

3. These boundary maps are quite non-trivial. For example, considering the central subgroup

\[ L_n^* = U_n^* := U^{n+1} \setminus U^n \subset U_n, \]

when the boundary map on

\[ H^1(G_w, [N_n^*(1) \times L_n^*(1)] \rtimes U_n) \]

is restricted to

\[ H^1(G_w, [N_n^*(1) \times L_n^*(1)] \rtimes L_n^n), \]
Non-abelian duality: remarks

3. These boundary maps are quite non-trivial. For example, considering the central subgroup

\[ L_n^n = U_n^n := U^{n+1} \setminus U^n \subset U_n, \]

when the boundary map on

\[ H^1(G_w, [N^*_n(1) \times L^*_n(1)] \rtimes U_n) \]

is restricted to

\[ H^1(G_w, [N^*_n(1) \times L^*_n(1)] \times L^n_n), \]

then it factors through

\[ H^1(G_w, N^*_n(1) \times (L^n_n)^*(1) \times L^n_n). \]
Non-abelian duality: remarks

On the subspace

\[ H^1(G_w, (L^n_n)^* (1) \times L^n_n) \subset H^1(G_w, N_n^* (1) \times (L^n_n)^* (1) \times L^n_n), \]

the induced map is usual Tate duality multiplied by \( n \).
Non-abelian duality: remarks

On the subspace

\[ H^1(G_w, (L_n^*)^*(1) \times L_n^n) \subset H^1(G_w, N_n^*(1) \times (L_n^n)^*(1) \times L_n^n), \]

the induced map is usual Tate duality multiplied by \( n \).

\[ H^1(G_w, (L_n^n)^*(1) \times L_n^n) \]

\[ H^1(G_w, [N_n^*(1) \times L_n^*(1)] \times L_n^n) \rightarrow H^1(G_w, N_n^*(1) \times (L_n^n)^*(1) \times L_n^n) \]

\[ H^1(G_w, [N_n^*(1) \times L_n^*(1)] \times U_n) \xrightarrow{\delta_{w,n}} \mathbb{Q}_p \]
Non-abelian duality: a reciprocity law

**Theorem**

The image of

\[ H^1(G_S, [N^*_n(1) \times L^*_n(1)] \rtimes U_n) \]

in

\[ \prod_{w \in S} H^1(G_w, [N^*_n(1) \times L^*_n(1)] \rtimes U_n) \]

is annihilated by

\[ \sum_w \delta_w. \]