A vanishing theorem for Fano varieties in positive characteristic

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1 Introduction

Let k be a perfect field of characteristic $p > 0$. $W = W(k)$ is the ring of Witt vectors of k and K is its fraction field.

By a *Fano variety* or an *anti-canonical variety* over k, we mean a smooth projective variety X over k such that the anti-canonical sheaf $(\Omega^d_X)^*$ is ample, where we denote by d the dimension of X.

In this note, we wish to prove the following

Theorem 1 *Let X be a Fano variety over k of dimension d. Then*

$$H^i(X, W\mathcal{O}_X) \otimes K = 0$$

for $i > 0$.

By Poincaré duality, this implies that

$$H^i(X, W\Omega^d_X) \otimes K = 0$$

for $i < d$, where $W\Omega^d_X$ is the sheaf of De Rham-Witt (DRW) differential forms of degree d constructed by Bloch and Illusie. This is because $H^i(W\mathcal{O}_X) \otimes K$ is the part of crystalline cohomology with Frobenius slopes in the interval $[0, 1[$, while $H^i(W, W\Omega^d_X)$ is the part with slope d ([9] cor. II.3.5). The author gave an erroneous proof of this corollary (in fact, an integral version) in an earlier preprint. H. Esnault [6] subsequently gave a correct proof using the Bloch-Srinivas decomposition theorem in rational Chow groups and rigid cohomology.

Here, we will use the additional structure provided by the De Rham-Witt complex ([2], [9], [7],[8]) in order to prove the theorem which is a slight strengthening of Esnault's result.

In fact, the theorem has the following corollary:

Corollary 1 *Let X be a Fano variety over a finite field with q elements. Then the number of rational points on X is congruent to 1 mod q.*

The corollary is an easy consequence of the Lefschetz trace formula for crystalline cohomology and slope arguments. Esnault's theorem gives that the number is congruent to 1 mod p, if $q = p^n$.

2 Proof

Throughout, if we write H^i without further embellishments, we mean rational crystalline cohomology.

We start with a quick summary of Esnault's proof (which was an adaptation of Bloch's proof in characteristic zero [3]): Because X is rationally connected and therefore $CH_0(X)\otimes Q = 0$ ([10]), one gets from the Bloch-Srinivas theorem [4] that the diagonal correspondence $\Delta \subset X \times X$ is equivalent in $CH_0(X)\otimes Q$ to a sum

$$z \times X + Z$$

where z is the class of a closed point and Z is a cycle supported on $X \times U$ for $U \subset X$ the complement of a divisor $D \subset X$. Therefore, if we apply the diagonal correspondence to a class of $H^i(X)$, $i > 0$, then the only thing that acts is the Z part. That is, if $i > 0$, and $\alpha \in H^i(X)$ then $[\Delta]_*(\alpha) = [Z]_*(\alpha)$. We have

$$H^i(X) \simeq H^i_{rig}(X)$$

where H^i_{rig} is Berthelot's rigid cohomology [1], and rigid cohomology has nice properties for open varieties. So if regard α as a class in $H^i_{rig}(X)$ and pull back to the subset U, then $\alpha_U = [Z_{X\times U}]_*(a) = 0$ since Z is supported on $X \times D$. On the other hand, one argues that the map $H^i_{rig}(X) \to H^i_{rig}(U)$ is injective on the Frobenius slope zero part. This concludes the argument.

Now we modify this proof. We wish to show that $H^i_{rig}(X) \to H^i_{rig}(U)$ is in fact injective on the part with slope in $[0, 1[$. First, let $f : Y \to X$ be an alteration [5] with the property that $E = f^*(D)$ is of normal crossing. We have a commutative diagram

$$\begin{array}{ccc} V & \hookrightarrow & Y \\ \downarrow & & \downarrow \\ U & \hookrightarrow & X \end{array}$$

where $V = Y - E$. Since the pullback from X to Y is injective, we just need to show that the map $H^i_{rig}(Y) \to H^i_{rig}(V)$ is injective on the part with slope in $[0, 1[$. But according to Shiho's comparison theorem [11], we have a commutative diagram

$$\begin{array}{ccc} H^i_{rig}(Y) & \to & H^i_{rig}(V) \\ \downarrow\simeq & & \downarrow\simeq \\ H^i(Y) & \to & H^i(Y, E) \end{array}$$

where the space $H^i(Y, E)$ refers to the rational log crystalline cohomology of the log scheme (Y, E). The map $H^i(Y) \to H^i(Y, E)$ is induced by a map

$$W\Omega_Y \to W\Omega_Y(\log E)$$

of De Rham-Witt complexes. This is because we can realize the map at the level of crystalline complexes [8] for the two log schemes Y (with trivial log structure) and (Y, E) and the De Rham-Witt complexes are just given level by level as the cohomology sheaves of the crystalline complexes. Meanwhile, the degree zero part is the same and equal to $W\mathcal{O}_Y$ for both complexes. So we are done if we can show that the slope spectral sequence for $H^i(Y,E)$ degenerates at E_1, as in the case without log structures, and induces an isomorphism between $H^i(W\Omega^j_Y(\log E))\otimes K$ and the part of $H^i(Y,E)$ with slope in $[j,j+1[$. To see this, one needs only repeat verbatim Bloch's argument from [2], III.3. This is because the log de Rham-Witt complex $W\Omega_Y(\log E)$ is also equipped with operators V and F satisfying

$$FV = VF = p$$

$$pFd = dF, \quad Vd = pdV$$

which is all that is necessary for Bloch's argument to apply: In brief, the map given by p^jF on $W\Omega^j_Y(\log E)$ is a map of complexes, and induces on log crystalline cohomology the action of the absolute Frobenius ϕ. So on each E_r of the spectral sequence, the map induced by p^jF on the subquotient E_r^{ji} of $H^i(W\Omega^j_Y(\log E))\otimes K$ is the map that commutes with the differentials. So the slope of ϕ on E_r^{ji} is $\geq j$. However, from $FV=p$ and the fact that V acts topologically nilpotently, we deduce that $\phi|E_r^{ji}$ has slope $< j+1$. Therefore, the difference of slopes forces all the differentials to be zero from E_1 on, and the Dieudonne-Manin classification of crystals allows us to split the filtration of the spectral sequence.

Let us dispense of the easy corollary 1: As already mentioned, $H^i(W\mathcal{O}_X)\otimes K$ is identified with the part of crystalline cohomology $H^i(X)$ on which the operator ϕ induced by the absolute Frobenius of X has slope in $[0,1[$. Thus, the vanishing shows that all the Frobenius slopes on $H^i(X)$ are ≥ 1 for $i>0$. Now when $q=p^n$ and k is the finite field $\mathbf{F}_q$, the Lefschetz trace formula gives us

$$|X(\mathbf{F}_q)| = \Sigma_i(-1)^i\mathrm{Tr}(\phi^n|H^i_{cr}(X)\otimes K)$$

which obviously yields our congruence.

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