

p -adic L -functions and Selmer varieties associated to elliptic curves with complex multiplication

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Abstract

We show how the finiteness of integral points on an elliptic curve over \mathbb{Q} with complex multiplication can be accounted for by the non-vanishing of L -functions that leads to bounds for dimensions of Selmer varieties.

The study of non-abelian fundamental groups renders it plausible that the principle of Birch and Swinnerton-Dyer, whereby non-vanishing of L -values, in some appropriate sense, accounts for the finiteness of integral points, can eventually be extended to hyperbolic curves. Here we will discuss the very simple case of a genus 1 hyperbolic curve X/\mathbb{Q} obtained by removing the origin from an elliptic curve E defined over \mathbb{Q} with complex multiplication by an imaginary quadratic field K . Denote by \mathcal{E} a Weierstrass minimal model of E and by \mathcal{X} the integral model of X obtained as the complement in \mathcal{E} of the closure of the origin. Let S be a set of primes including the infinite place and those of bad-reduction for \mathcal{E} . We wish to examine the theorem of Siegel, asserting the finiteness of $\mathcal{X}(\mathbb{Z}_S)$, the S -integral points of \mathcal{X} , from the point of view of fundamental groups and Selmer varieties. In particular, we show how the finiteness of points can be proved using ‘the method of Coates and Wiles’ which, in essence, makes use of the non-vanishing of p -adic L -functions arising from the situation.

That is to say, in studying the set $\mathcal{E}(\mathbb{Z}_S)(= \mathcal{E}(\mathbb{Z}) = E(\mathbb{Q}))$, Coates and Wiles showed the special case of the conjecture of Birch and Swinnerton-Dyer by deriving the finiteness of $\mathcal{E}(\mathbb{Z}_S)$ from the non-vanishing of $L(E/\mathbb{Q}, s)$ at $s = 1$. Of course the L -function can vanish at 1 in general, in which case $\mathcal{E}(\mathbb{Z}_S)$ is supposed to be infinite. But we know that $\mathcal{X}(\mathbb{Z}_S)$ is always finite. From the perspective of this paper, this is a consequence of the fact that appropriate p -adic L -functions have only finitely many zeros. More precisely, if we choose a prime p that splits as $p = \pi\bar{\pi}$ in K and let \mathcal{L} and $\bar{\mathcal{L}}$ denote the p -adic L -functions associated to the π and $\bar{\pi}$ -power torsion points of E/K [7], we know that they have only finitely many zeros. And then, the non-vanishing of L -functions forces the vanishing of infinitely many \mathbb{Q}_p -Selmer groups for a family of Galois representations naturally associated to X . The motivic tool used to put this information together in the present approach is a natural quotient

W

of the \mathbb{Q}_p -pro-unipotent fundamental group U of X with base point at an S -integral point b (which we assume to exist) and, as usual, its further quotients W_n modulo the descending central series. Integral points of \mathcal{X} give rise to torsors for the W_n that are classified by a projective system of global Selmer varieties

$$H_f^1(\Gamma, W_n)$$

where $\Gamma = \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ is the Galois group over \mathbb{Q} of an algebraic closure. That is, there is a diagram

$$\begin{array}{ccc} \mathcal{X}(\mathbb{Z}_S) & \hookrightarrow & \mathcal{X}(\mathbb{Z}_p) \\ \downarrow & & \downarrow \\ H_f^1(\Gamma, W_n) & \rightarrow & H_f^1(\Gamma_p, W_n) \end{array}$$

obtained from the formalism of the fundamental group that associates to each point x the U -torsor of paths from b to x and then pushes it out to a W_n -torsor. Here, $\Gamma_p = \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ embedded into Γ as a decomposition group at p .

Let $G = \text{Gal}(K(E[\pi^\infty])/K)$ and $\bar{G} = \text{Gal}(K(E[\bar{\pi}^\infty])/K)$, and let $\Lambda = \mathbb{Z}_p[[G]]$, $\bar{\Lambda} = \mathbb{Z}_p[[\bar{G}]]$ be the corresponding Iwasawa algebras, so that $\mathcal{L} \in \Lambda$ and $\bar{\mathcal{L}} \in \bar{\Lambda}$. Denote by

$$\chi : \Lambda \rightarrow \mathbb{Q}_p, \quad \bar{\chi} : \bar{\Lambda} \rightarrow \mathbb{Q}_p$$

the homomorphisms corresponding to the Galois actions on $V_\pi(E) = (\varprojlim E[\pi^n]) \otimes \mathbb{Q}$ and $V_{\bar{\pi}}(E) = (\varprojlim E[\bar{\pi}^n]) \otimes \mathbb{Q}$. Finally, let $r = \dim H_f^1(\Gamma, V_p(E))$ and $s = |S|$.

Theorem 0.1 *We have the inequality of dimensions*

$$\dim H_f^1(\Gamma, W_n) < \dim H_f^1(\Gamma_p, W_n)$$

for all n sufficiently large.

The fact that the p -adic L -functions can have at most finitely many zeros is exactly the required input for this theorem. But the location of the zeros, of course, is a highly non-trivial issue. On the other hand,

Theorem 0.2 *Suppose $\chi^k(\mathcal{L}) \neq 0$ and $\bar{\chi}^k(\bar{\mathcal{L}}) \neq 0$ for each $k < 0$. Then*

$$\dim H_f^1(\Gamma, W_n) < \dim H_f^1(\Gamma_p, W_n)$$

for all $n \geq r + s + 1$.

I am informed by John Coates that the non-vanishing in the hypothesis is a conjecture appearing in folklore.

The way the implication works out is that each non-vanishing of an L -value implies the vanishing of some H^2 in Galois cohomology, and W is constructed so as to avoid the complications that arise when working directly with U . The finiteness of global points then follows in a straightforward way as in previous work (e.g [4]) whereby the inequality implies the existence of certain p -adic iterated integrals that vanish on the global points. Since we hope the method will eventually lead to a direct construction of a p -adic analytic function that annihilates the global points, the more refined statement of the second theorem seems worth making explicit. Of course in the present work the main emphasis is the sequence of implications

Non-vanishing of L -values \Rightarrow control of Selmer varieties \Rightarrow finiteness of global points,

entirely parallel to the case of rational points on compact elliptic curves, with just the replacement of Selmer groups by Selmer varieties.

A word of caution regarding the notation: At the urging of Richard Hain, the indexing of the finite-dimensional quotients of U has been shifted. So our U_n is U_{n+1} from the papers [4] and [6], for example. However, the scheme here is consistent with that of [3].

1 A quotient of the unipotent fundamental group

The quotient in question is constructed as follows.

Let $U = \pi_1^{un}(\bar{X}, b)$ be the \mathbb{Q}_p -pro-unipotent completion of $\hat{\pi}_1(\bar{X}, b)$ (see [2]) and let U^n denote the descending central series, normalized so that $U^1 = U$. Define $U_n = U^{n+1} \setminus U$. We then have exact sequences

$$0 \rightarrow U^{n+1} \setminus U^n \rightarrow U_n \rightarrow U_{n-1} \rightarrow 0$$

for $n \geq 1$. Denote by L the Lie algebra of U with descending central series L^n . Thus, we have natural isomorphisms

$$U^{n+1} \backslash U^n \simeq L^{n+1} \backslash L^n$$

compatible with the action of Γ . Since $\hat{\pi}_1(\bar{X}, b)$ is pro-finite free on two generators, L is the pro-nilpotent completion of the free Lie algebra on two generators, where the generators can be any two elements projecting to a basis of $L_1 = H_1(\bar{X}, \mathbb{Q}_p)$. Therefore, L comes with a natural grading (not compatible with the Galois action)

$$L = \overline{\bigoplus_{n=1}^{\infty} L(n)}$$

where $L(n)$ is generated by the Lie monomials of degree n in the generators [8]. On the other, in the current situation, we have

$$L_1 = V_p(E) \simeq V_{\pi}(E) \oplus V_{\bar{\pi}}(E)$$

so that L even has a bi-grading

$$L = \overline{\bigoplus L_{i,j}}$$

That is, taking e and f to be elements in L_1 that map to bases of V_{π} and $V_{\bar{\pi}}$ respectively, $L_{i,j}$ is spanned by Lie monomials that have i number of e 's and j number of f 's, e.g.,

$$ad^{i-1}(e)(ad^{j-1}(f)([e, f]))$$

This bi-grading also induces a filtration

$$L_{\geq n, \geq m} := \overline{\bigoplus_{i \geq n, j \geq m} L_{i,j}}$$

by Lie ideals. Let $N = \text{Gal}(\bar{\mathbb{Q}}/K) \subset \Gamma$ so that $\Gamma = N\sigma$, where σ is complex conjugation. Then if $x \in N$, we have

$$xe = \chi(x)e + z$$

and

$$xf = \bar{\chi}(x)f + z'$$

for $z, z' \in L^2$. But $L^2 \subset L_{\geq 1, \geq 1}$. Hence, if $l \in L_{i,j}$, then an easy induction shows that

$$xl = \chi(x)^i \bar{\chi}(x)^j l + z$$

for $z \in L_{\geq i+1, \geq j+1}$. In particular, each $L_{\geq n, \geq m}$ is stabilized by N . Similarly,

$$\sigma(L_{\geq i, \geq j}) \in L_{\geq j, \geq i}$$

so we see that

$$L_{\geq n, \geq n}$$

is stabilized by Γ for each n . Therefore, we get a quotient Lie algebra

$$L \rightarrow \mathcal{W} := L/L_{\geq 2, \geq 2} \rightarrow 0$$

and a corresponding quotient group

$$U \rightarrow W \rightarrow 0$$

with a compatible Γ -action. If we choose the ordering $e < f$, then $[e, f]$ is a Hall basis [8] for $L^3 \backslash L^2$, and inside the Hall basis for $L^{n+1} \backslash L^n$, $n \geq 3$, are the elements

$$ad^{n-2}(e)([e, f])$$

and

$$ad^{n-2}(f)([e, f]).$$

All the other basis elements are clearly in $L_{\geq 2, \geq 2}$. Thus,

$$W^{n+1} \backslash W^n \simeq \mathcal{W}^{n+1} \backslash \mathcal{W}^n$$

is generated by the class of $ad^{n-2}(e)([e, f])$ and $ad^{n-2}(f)([e, f])$. That is, as Γ -modules, we have

$$W^{n+1} \backslash W^n \simeq \mathbb{Q}_p(\chi^{n-2}(1)) \oplus \mathbb{Q}_p(\bar{\chi}^{n-2}(1))$$

for $n \geq 2$, where σ acts by sending the generator $ad^{n-2}(e)([e, f])$ to $-ad^{n-2}(f)([e, f])$.

2 The unipotent Albanese map

There are various ways to see that W is unramified outside S and crystalline at p . For example, by construction, the coordinate ring of W is a sub-ring of that of U , and hence, the conditions of being unramified or crystalline ([4], section 2) are inherited from U .

Now we examine the unipotent Albanese map [4]

$$\mathcal{X}(\mathbb{Z}_S) \rightarrow H^1(\Gamma, U_n),$$

obtained by associating to $x \in \mathcal{X}(\mathbb{Z}_S)$ the class of the U_n -torsor

$$P(x) := \pi_1^{un}(\bar{X}; b, x)$$

of unipotent paths from b to x . We will continue this map to

$$\mathcal{X}(\mathbb{Z}_S) \rightarrow H^1(\Gamma, U_n) \rightarrow H^1(\Gamma, W_n)$$

obtained by composing with the quotient map. At the level of torsors, the map

$$H^1(\Gamma, U_n) \rightarrow H^1(\Gamma, W_n)$$

is given by the pushout:

$$Z \mapsto (Z \times W_n)/U_n$$

Since the condition of being crystalline at p ([4], section 2) merely signifies that a torsor has a Γ_p -invariant B_{cr} point, this condition is clearly preserved by push-out. Thus, we have an induced map

$$H_f^1(\Gamma_p, U_n) \rightarrow H_f^1(\Gamma_p, W_n)$$

where the subscript f refers exactly to the subset

$$H_f^1(\Gamma_p, W_n) \subset H^1(\Gamma_p, W_n)$$

of cohomology classes that are crystalline. Similarly, the condition of being unramified at $v \notin T = S \cup \{p\}$ is preserved under pushout. Let Γ_T denote the Galois group of the maximal extension of \mathbb{Q} unramified outside T . By [3], the system

$$H^1(\Gamma_T, W_n)$$

has the structure of a pro-algebraic variety over \mathbb{Q}_p , as does the sub-system

$$H_f^1(\Gamma_T, W_n) \subset H^1(\Gamma_T, W_n)$$

classifying torsors that are unramified outside T and crystalline at p . Thus, we also have an induced map of global Selmer varieties

$$H_f^1(\Gamma, U_n) \rightarrow H_f^1(\Gamma, W_n)$$

giving rise to a commutative diagram

$$\begin{array}{ccc} \mathcal{X}(\mathbb{Z}_S) & \rightarrow & \mathcal{X}(\mathbb{Z}_p) \\ \downarrow & & \downarrow \\ H_f^1(\Gamma, W_n) & \rightarrow & H_f^1(\Gamma_p, W_n) \end{array}$$

Since each map

$$\mathcal{X}(\mathbb{Z}_p) \rightarrow H_f^1(\Gamma_p, U_n)$$

has Zariski dense image, so do the maps

$$\mathcal{X}(\mathbb{Z}_p) \rightarrow H_f^1(\Gamma_p, W_n).$$

As in [3] and [4], this denseness is an important ingredient in extracting Diophantine finiteness out of the theorems.

3 Proof of the theorems

The proof is now straightforward. Recall that we have a sequence

$$0 \rightarrow H^1(\Gamma_T, W^{n+1} \setminus W^n) \rightarrow H^1(\Gamma_T, W_n) \rightarrow H^1(\Gamma_T, W_{n-1})$$

which is exact in that the vector group kernel acts on the middle term with quotient variety being the image of the second map. Furthermore, if we examine

$$H_f^1(\Gamma_p, W^{n+1} \setminus W^n) \simeq H_f^1(\Gamma_p, \mathbb{Q}_p(\chi^{n-2}(1)) \oplus \mathbb{Q}_p(\bar{\chi}^{n-2}(1)))$$

we see that all classes are crystalline for $n \geq 3$. This is because, for example, the crystalline classes in $H^1(\Gamma_p, \mathbb{Q}_p(\chi^{n-2}(1)))$ are classified by

$$D^{DR}(\mathbb{Q}_p(\chi^{n-2}(1)))/F^0$$

where $D^{DR}(\cdot) = [(\cdot) \otimes B_{DR}]^{\Gamma_p}$ is Fontaine's Dieudonné functor [1]. But

$$D^{DR}(\mathbb{Q}_p(\chi^{n-2}(1))) = D^{DR}(\mathbb{Q}_p(\chi^{n-2}))(1)$$

and the Hodge filtration on $D^{DR}(\mathbb{Q}_p(\chi^{n-2}))$ is non-positive. Therefore, the Hodge filtration on $D^{DR}(\mathbb{Q}_p(\chi^{n-2}(1)))$ is strictly negative. Then a simple dimension count shows that

$$H_f^1(\Gamma_p, \mathbb{Q}_p(\chi^{n-2}(1))) = H^1(\Gamma_p, \mathbb{Q}_p(\chi^{n-2}(1)))$$

for $n \geq 3$ and the previous exact sequence can be re-written

$$0 \rightarrow H^1(\Gamma_T, W^{n+1} \setminus W^n) \rightarrow H_f^1(\Gamma_T, W_n) \rightarrow H_f^1(\Gamma_T, W_{n-1})$$

for $n \geq 3$. For $n = 2$, we have

$$0 \rightarrow H_f^1(\Gamma_T, \mathbb{Q}_p(1)) \rightarrow H_f^1(\Gamma_T, W_2) \rightarrow H_f^1(\Gamma_T, W_1)$$

and

$$H_f^1(\Gamma_T, \mathbb{Q}_p(1)) \simeq (\mathbb{Z}_S^*) \otimes \mathbb{Q}_p$$

(as in [6], section 2) so that

$$\dim H_f^1(\Gamma_T, W_2) \leq r + s - 1$$

If we put this together, we get

$$\dim H_f^1(\Gamma_T, W_n) \leq r + s - 1 + \sum_{i=3}^n \dim H^1(\Gamma_T, W^{i+1} \setminus W^i)$$

for $n \geq 3$. As for the dimensions of the intervening H^1 's, we have the Euler characteristic formula

$$\dim H^1(\Gamma_T, W^{i+1} \setminus W^i) = \dim H^2(\Gamma_T, W^{i+1} \setminus W^i) + \dim(W^{i+1} \setminus W^i)^-$$

where the superscript refers to the subspace where σ acts as (-1) . But σ exchanges the one-dimensional factors of $\mathbb{Q}_p(\chi^{n-2}(1)) \oplus \mathbb{Q}_p(\bar{\chi}^{n-2}(1))$ so that $\dim(W^{i+1} \setminus W^i)^- = 1$.

Meanwhile, for the local cohomologies, we can calculate the dimensions explicitly. Firstly, we know that

$$\dim H_f^1(\Gamma_p, W_2) = \dim H_f^1(\Gamma_p, U_2) = 2$$

([4], section 4) On the other hand, $H^2(\Gamma_p, W^{n+1} \setminus W^n) = 0$ for $n \geq 3$ so that the map

$$H^1(\Gamma_p, W_{n+1}) \rightarrow H^1(\Gamma_p, W_n)$$

is surjective for $n \geq 3$. As remarked above, we also have $H_f^1(\Gamma_p, W^{n+1} \setminus W^n) = H^1(\Gamma_p, W^{n+1} \setminus W^n)$ for $n \geq 3$. This implies that we have an exact sequence

$$0 \rightarrow H_f^1(\Gamma_p, W^{n+1} \setminus W^n) \rightarrow H_f^1(\Gamma_p, W_n) \rightarrow H_f^1(\Gamma_p, W_{n-1}) \rightarrow 0$$

for $n \geq 3$, where each $H_f^1(\Gamma_p, W^{n+1} \setminus W^n)$ has dimension 2. So

$$\dim H_f^1(\Gamma_p, W_n) = 2(n-2) + 2 = 2n - 2$$

for $n \geq 2$.

It remains to prove the

Claim 3.1

$$H^2(\Gamma_T, W^{n+1} \setminus W^n) = 0$$

for n sufficiently large.

and

Claim 3.2 *If $\chi^k(\Lambda) \neq 0$ and $\bar{\chi}^k(\bar{\Lambda}) \neq 0$ for $k < 0$, then*

$$H^2(\Gamma_T, W^{n+1} \setminus W^n) = 0$$

for $n \geq 3$.

Since 3.1 implies that

$$\dim H^1(\Gamma_T, W^{n+1} \setminus W^n) = 1$$

for n sufficiently large, we see that there is a constant such that

$$H_f^1(\Gamma_T, W_n) = C + n$$

while the local dimensions grow like $2n$. Hence we get the statement of theorem 0.1.

On the other hand, with 3.2, we get

$$\dim H_f^1(\Gamma_T, W_n) \leq r + s + n - 2$$

so that we get the desired inequality of dimensions as soon as

$$r + s + n - 2 < 2n - 2$$

or $n \geq r + s + 1$.

We proceed to prove the claims. Clearly it suffices to consider the Galois cohomology of $N_T \subset G_T$, where N_T is the Galois group of the maximal extension of K unramified outside the primes dividing T . For any continuous representation M of N_T , we define the kernel Sha^i of the localization maps on cohomology as

$$0 \rightarrow Sha^i(M) \rightarrow H^i(N_T, M) \rightarrow \bigoplus_{v|T} H^i(N_v, M)$$

where $N_v \subset N_T$ is a decomposition group for the prime v . So we have

$$0 \rightarrow Sha^2(W^{n+1} \setminus W^n) \rightarrow H^2(\Gamma_T, W^{n+1} \setminus W^n) \rightarrow \bigoplus_{v|T} H^2(N_v, W^{n+1} \setminus W^n)$$

By local duality,

$$\begin{aligned} H^2(N_v, W^{n+1} \setminus W^n) &= H^2(N_v, \mathbb{Q}_p(\chi^{n-2}(1)) \oplus \mathbb{Q}_p(\bar{\chi}^{n-2}(1))) \\ &\simeq H^0(N_v, \mathbb{Q}_p(\chi^{2-n}) \oplus \mathbb{Q}_p(\bar{\chi}^{2-n}))^* = 0 \end{aligned}$$

for $n \geq 3$ from which we get

$$Sha^2(W^{n+1} \setminus W^n) \simeq H^2(N_T, W^{n+1} \setminus W^n).$$

By Poitou-Tate duality, we have

$$Sha^2(W^{n+1} \setminus W^n) \simeq Sha^1((W^{n+1} \setminus W^n)^*(1))^* \simeq Sha^1(\mathbb{Q}_p(\chi^{2-n}))^* \oplus Sha^1(\mathbb{Q}_p(\bar{\chi}^{2-n}))^*$$

But using the inflation-restriction exact sequence, we get

$$Sha^1(\mathbb{Q}_p(\chi^{2-n})) \simeq \text{Hom}_\Lambda(A \otimes \mathbb{Q}, \mathbb{Q}_p(\chi^{2-n}))$$

where A is the Galois group of the maximal unramified pro- p extension of $K(E[\pi^\infty])$ split above the primes dividing T . Now, by [7], we know that the p -adic L -function \mathcal{L} annihilates $A \otimes \mathbb{Q}$. On the other hand, Λ acts on $\mathbb{Q}_p(\chi^{2-n})$ through the character χ^{2-n} and we know $\chi^{2-n}(\mathcal{L}) \neq 0$ for n sufficiently large. Therefore,

$$\text{Hom}_\Lambda(A \otimes \mathbb{Q}, \mathbb{Q}_p(\chi^{2-n})) = 0$$

for $n \gg 0$. There is a parallel argument for $Sha^1(\mathbb{Q}_p(\bar{\chi}^{2-n}))$ which then yields Claim 3.1. For Claim 3.2, the argument is exactly the same, except that the refined non-vanishing hypothesis implies

$$Sha^1(\mathbb{Q}_p(\chi^{2-n})) \oplus Sha^1(\mathbb{Q}_p(\bar{\chi}^{2-n})) = 0$$

for $n \geq 3$.

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