December 14, 2007

Hi John,

Thanks for persisting with the question on $X = \mathrm{Spec}(\mathbb{Z})$ and three-manifolds. It does seem now that the question is quite serious, and I didn't do it proper justice when we spoke in September. Perhaps I'll be able to say something substantive at some later time, but I thought I'd just make a few elementary points that may be helpful. We will have to discuss some cohomology, but nothing beyond the intuition from usual topology (of manifolds).

In particular, while I do agree with the analogy made by James, one objection is that it seems like having a ground field of some sort, which for X would be the mythical field $\mathbb{F}_1$ with one element, is essential to the three-dimensional nature.

It's not, in my opinion. James is taking the approach of counting dimensions of a fiber. But I would rather count the dimension of a normal bundle, thereby eliminating the need for a base field. That is, I think we all agree by now that $x_p := \mathrm{Spec}(\mathbb{F}_p)$ has dimension one because it's a $K(\hat{\mathbb{Z}}, 1)$. Now consider its embedding

$$x_p \hookrightarrow X$$

Then the observation is that the normal bundle of this embedding is two-dimensional, and hence, X is three-dimensional.

We can make this a bit more precise by discussing cohomology with coefficient in the sheaf of p^n-the roots of unity, which I'll denote by μ, with some p being understood. I'll also assume p odd, because there's a certain definition I can't remember if p allowed to be 2. μ plays the role of an orientation sheaf in étale cohomology. Let

$$U = X \setminus \{x_p\} = \mathrm{Spec}(\mathbb{Z}[1/p]).$$

By using compact support cohomology, we can compare the cohomology of X and U via a long exact sequence

$$\cdots \to H^i_c(U, \mu) \to H^i(X, \mu) \to H^i(x_p, \mu) \to \cdots$$

where we have noted that for X, compact support cohomology is the same as usual cohomology because X is 'compact'. Actually, we need to add the contribution of the Archimedean prime to get a cohomologically compact space, but for $p \neq 2$ it doesn't matter. (The Archimedean prime doesn't contribute anything anyways, because $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ has order 2.) Now, since x_p has cohomological dimension 1, we see that

$$H^i_c(U, \mu) \simeq H^i(X, \mu)$$

for $i \geq 3$. But there is another way to compute $H^i_c(U, \mu)$, namely, by comparing it with the cohomology $H^i(U, \mu)$ *without* the compact support condition. In that case, what we get is a long exact sequence

$$\cdots \to H^i_c(U, \mu) \to H^i(U, \mu) \to H^i(D^*, \mu) \to \cdots$$

where $D^* = \mathrm{Spec}(\mathbb{Q}_p)$. The way to think about this sequence is that D^* is the punctured disk bundle obtained from $D = \mathrm{Spec}(\mathbb{Z}_p)$ by removing x_p. Note the important point here that I wrote 'punctured disk bundle' rather than 'punctured disk.' This is in view of the fact that x_p is 'one-dimensional.' Notice also that $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n$ where p is the defining equation for x_p inside X. So $\mathbb{Z}_p$ consists of the functions defined on an infinitesimal neighborhood of x_p which would be homotopic to a tubular neighborhood T in topology. (Actually, the general rule is that when you take this kind of completion near a regular subscheme defined by an ideal I, it's like a twist of the normal bundle. For example, there is a filtration of the completed ring whose associated graded ring is $\mathrm{Sym}(I/I^2)$. But the twist is such that it still should be homotopic to the normal bundle, or a tubular neighborhood, because the filtration is something that would split in a more flabby situation.) In any case, D^* is then homotopic to the boundary $\partial T$of a tubular neighborhood of x_p, or the sphere bundle of the normal bundle. So the above sequence can be identified with the sequence

$$\cdots \to H^i_c(U, \mu) \to H^i(\bar{U}, \mu) \to H^i(\partial T, \mu) \to \cdots$$

familiar from topology, where $\bar{U}$ is the manifold with boundary obtained by removing the interior of T from X. But the point is that now, $H^i(D^*, \mu)$ has the potential to contribute to $H^{i+1}_c(U, \mu)$, and it does for $i = 2$! That is to say, the map

$$H^2(U, \mu) \to H^2(D^*, \mu)$$

is always zero because $H^2(U, \mu) = 0$. This follows from the fact that a central simple algebra cannot be non-split at just one prime. Meanwhile, $H^3(U, \mu) = 0$ so that

$$H^2(D^*, \mu) \simeq H^3_c(U, \mu) \simeq H^3(X, \mu).$$

(The H^3 vanishing assertion has to do with the fact that x_p has been removed, so that U has no p among its 'residue characteristics,' while μ is p-power torsion. Within the class of such sheaves, U acts like dimension 2.) As it turns out, $H^2(D^*) \simeq \mathbb{Z}/p^n$. This is a fact from local class field theory. But it's not hard to see an intuitive reason: D^* is homotopic to a circle bundle over a circle (x_p). So it all boils down to the fact that x_p has dimension 1. In conclusion, $H^3(X, \mu) = \mathbb{Z}/p^n$ exactly as for a compact three-manifold. It's interesting that the contribution to $H^3(X, \mu)$ comes just from $\mathrm{Spec}(\mathbb{Q}_p)$ for the given prime p.

Besides this outline, another general fact about counting dimensions that's often taken for granted is that Krull codimension 1 of x_p in D (or X) should count as 'real codimension 2.' One way to justify this (besides the justification that you might say is implicit in the argument above) is by considering unramified coverings of D^*. By taking roots of the coordinate function p, we get a tower of non-trivial coverings. This is the sense in which D^* is much more like a punctured plane ($\mathbb{C}^*$) than a punctured line ($\mathbb{R}^*$). On the punctured line the coordinate function x has a unique m-th root for any odd m. This is not true for the coordinate function x on $\mathbb{C}^*$ or the coordinate function p on $\mathrm{Spec}(\mathbb{Q}_p)$, and this is manifested in the coverings you get of the two latter by extracting roots, and eventually contributes to the cohomological similarities.

Finally, regarding the field with one element. I'm all for general theory building, but I think this is one area where having some definite problems in mind might help to focus ideas better. From this perspective, there are two things to look for in the theory of $\mathbb{F}_1$.

(1) A theory of differentiation with respect to the ground field. A well-known consequence of such a theory could include an array of effective theorems in Diophantine geometry, like an effective Mordell conjecture or the ABC conjecture. Over function fields, the ability to differentiate with respect to the field of constants is responsible for the considerably stronger theorems of Mordell conjecture type, and makes the ABC conjecture trivial.

(2) A good notion of a fiber product

$$\mathrm{Spec}(\mathbb{Z}) \times_{\mathrm{Spec}(\mathbb{F}_1)} \mathrm{Spec}(\mathbb{Z})$$

and a diagonal Δ inside it. There are a number of approaches to the Riemann hypothesis for curves over finite fields that consist of an analysis of this diagonal, that is, of how it intersects with the graph of Frobenius morphisms. It would be nice if some theorist could at least provide a rigorous argument that the RH over $\mathbb{Q}$ follows from a reasonable collection of properties of $\mathbb{F}_1$ and the map $\mathrm{Spec}(\mathbb{Z}) \to \mathrm{Spec}(\mathbb{F}_1)$.

I guess I've become somewhat more pragmatic in recent years. But even in something much better established, such as the theory of motives, it's sometimes exasperating that there are very few papers that try to use the theory to prove something definite outside the area.

Best,

Minhyong