4.2 Directional Derivative

For a function of 2 variables $f(x, y)$, we have seen that the function can be used to represent the surface

$$z = f(x, y)$$

and recall the geometric interpretation of the partials:

(i) $f_x(a, b)$-represents the rate of change of the function $f(x, y)$ as we vary $x$ and hold $y = b$ fixed.

(ii) $f_y(a, b)$-represents the rate of change of the function $f(x, y)$ as we vary $y$ and hold $x = a$ fixed.

We now ask, at a point $P$ can we calculate the slope of $f$ in an arbitrary direction?

Recall the definition of the vector function $\nabla f$,

$$\nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right).$$

We observe that,

$$\nabla f \cdot \hat{i} = f_x$$
$$\nabla f \cdot \hat{j} = f_y$$

This enables us to calculate the directional derivative in an arbitrary direction, by taking the dot product of $\nabla f$ with a unit vector, $\vec{u}$, in the desired direction.

**Definition.** The directional derivative of the function $f$ in the direction $\vec{u}$ denoted by $D_{\vec{u}}f$, is defined to be,

$$D_{\vec{u}}f = \frac{\nabla f \cdot \vec{u}}{|\vec{u}|}$$

**Example.** What is the directional derivative of $f(x, y) = x^2 + xy$, in the direction $\hat{i} + 2\hat{j}$ at the point $(1, 1)$?
Solution: We first find $\nabla f$.

$$\nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$$

$$= (2x + y, x)$$

$\nabla f(1, 1) = (3, 1)$

Let $u = \vec{i} + 2\vec{j}$.

$$|\vec{u}| = \sqrt{1^2 + 2^2} = \sqrt{1 + 4} = \sqrt{5}.$$

$$D_{\vec{u}} f(1, 1) = \frac{\nabla f \cdot \vec{u}}{|\vec{u}|}$$

$$= \frac{(3, 1). (1, 2)}{\sqrt{5}}$$

$$= \frac{(3)(1) + (1)(2)}{\sqrt{5}}$$

$$= \frac{5}{\sqrt{5}}$$

$$= \sqrt{5}$$

Properties of the Gradient deduced from the formula of Directional Derivatives

$$D_{\vec{u}} f = \frac{\nabla f \cdot \vec{u}}{|\vec{u}|}$$

$$= \frac{|\nabla f||\vec{u}| \cos \theta}{|\vec{u}|}$$

$$= |\nabla f| \cos \theta$$

1. If $\theta = 0$, i.e. i.e. $\vec{u}$ points in the same direction as $\nabla f$, then $D_{\vec{u}} f$ is maximum. Therefore we may conclude that

(i) $\nabla f$ points in the steepest direction.

(ii) The magnitude of $\nabla f$ gives the slope in the steepest direction.
2. At any point \( P \), \( \nabla f(P) \) is \textbf{perpendicular to the level set} through that point.

**Example.** 1. Let \( f(x, y) = x^2 + y^2 \) and let \( P = (1, 2, 5) \). Then \( P \) lies on the graph of \( f \) since \( f(1, 2) = 5 \). Find the slope and the direction of the steepest ascent at \( P \) on the graph of \( f \).

**Solution:**  
- We use the first property of the Gradient vector. The direction of the steepest ascent at \( P \) on the graph of \( f \) is the direction of the gradient vector at the point \( (1, 2) \).

\[
\nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = (2x, 2y)
\]

\[
\nabla f(1, 2) = (2, 4).
\]

- The slope of the steepest ascent at \( P \) on the graph of \( f \) is the magnitude of the gradient vector at the point \( (1, 2) \).

\[
|\nabla f(1, 2)| = \sqrt{2^2 + 4^2} = \sqrt{20}.
\]

2. Find a normal vector to the graph of the equation \( f(x, y) = x^2 + y^2 \) at the point \( (1, 2, 5) \). Hence write an equation for the tangent plane at the point \( (1, 2, 5) \).

**Solution:** We use the second property of the gradient vector. For a function \( g \), \( \nabla g(P) \) is \textbf{perpendicular to the level set}. So we want our surface \( z = x^2 + y^2 \) to be the level set of a function.

Therefore we define a new function, \( g(x, y, z) = x^2 + y^2 - z \).

Then our surface is the level set

\[
\begin{align*}
g(x, y, z) &= 0 \\
x^2 + y^2 - z &= 0 \\
z &= x^2 + y^2
\end{align*}
\]
\[ \nabla g = \left( \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right) \]
\[ = (2x, 2y, -1) \]
\[ \nabla g(1, 2, 5) = (2, 4, -1) \]

By the above property, \( \nabla g(P) \) is perpendicular to the level set \( g(x, y, z) = 0 \). Therefore \( \nabla g(P) \) is the required normal vector.

Finally an equation for the tangent plane at the point \((1, 2, 5)\) on the surface is given by

\[ 2(x - 1) + 4(y - 2) - 1(z - 5) = 0. \]

### 4.3 Curl and Divergence

We denoted the gradient of a scalar function \( f(x, y, z) \) as

\[ \nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \]

Let us separate or isolate the operator \( \nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \). We can then define various physical quantities such as div, curl by specifying the action of the operator \( \nabla \).

#### Divergence

**Definition.** Given a vector field \( \vec{v}(x, y, z) = (v_1(x, y, z), v_2(x, y, z), v_3(x, y, z)) \), the divergence of \( \vec{v} \) is a scalar function defined as the dot product of the vector operator \( \nabla \) and \( \vec{v} \),

\[ \text{Div} \ \vec{v} = \nabla \cdot \vec{v} = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (v_1, v_2, v_3) = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \]

**Example.** Compute the divergence of \((x - y)\hat{i} + (x + y)\hat{j} + z\hat{k} \).
Solution:

\[ \vec{v} = ((x - y), (x + y), z) \]
\[ \nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \]
\[ \text{Div } \vec{v} = \nabla \cdot \vec{v} = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot ((x - y), (x + y), z) \]
\[ = \frac{\partial(x - y)}{\partial x} + \frac{\partial(x + y)}{\partial y} + \frac{\partial z}{\partial z} \]
\[ = 1 + 1 + 1 \]
\[ = 3 \]

Curl

**Definition.** The curl of a vector field is a vector function defined as the cross product of the vector operator \( \nabla \) and \( \vec{v} \),

\[ \text{Curl } \vec{v} = \nabla \times \vec{v} = \left| \begin{array}{ccc} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{array} \right| \]
\[ = \left( \frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \right)i - \left( \frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z} \right)j + \left( \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right)k \]

**Example.** Compute the curl of the vector function \((x - y)i + (x + y)j + zk\).

**Solution:**

\[ \text{Curl } \vec{v} = \nabla \times \vec{v} = \left| \begin{array}{ccc} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x - y) & (x + y) & z \end{array} \right| \]
\[ = \left( \frac{\partial z}{\partial y} - \frac{\partial(x + y)}{\partial z} \right)i - \left( \frac{\partial z}{\partial x} - \frac{\partial(x - y)}{\partial z} \right)j + \left( \frac{\partial(x + y)}{\partial x} - \frac{\partial(x - y)}{\partial y} \right)k \]
\[ = (0 - 0)i - (0 - 0)j + (1 - (-1))k \]
\[ = 2k \]
4.4 Laplacian

We have seen above that given a vector function, we can calculate the divergence and curl of that function. A scalar function \( f \) has a vector function \( \nabla f \) associated to it. We now look at \( \text{Curl}(\nabla f) \) and \( \text{Div}(\nabla f) \).

\[
\text{Curl}(\nabla f) = \nabla \times \nabla f = (\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z})i + (\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x})j + (\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y})k
\]

\[= (f_{yz} - f_{zy})i + (f_{xz} - f_{zx})j + (f_{xy} - f_{yx})k \]

\[= 0 \]

\[
\text{Div}(\nabla f) = \nabla \cdot \nabla f = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)
\]

\[= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \]

**Definition.** The Laplacian of a scalar function \( f(x, y) \) of two variables is defined to be \( \text{Div}(\nabla f) \) and is denoted by \( \nabla^2 f \),

\[\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.\]

The Laplacian of a scalar function \( f(x, y, z) \) of three variables is defined to be \( \text{Div}(\nabla f) \) and is denoted by \( \nabla^2 f \),

\[\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.\]

**Example.** Compute the Laplacian of \( f(x, y, z) = x^2 + y^2 + z^2 \).

**Solution:**

\[
\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}
\]

\[= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}
\]

\[= 2 + 2 + 2
\]

\[= 6.\]
We have the following identities for the Laplacian in different coordinate systems:

**Rectangular**: \[ \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \]

**Polar**:
\[ \nabla^2 f = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} \]

**Cylindrical**:
\[ \nabla^2 f = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} \]

**Spherical**:
\[ \nabla^2 f = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{\rho^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2} \]

**Example.** Consider the same function \( f(x, y, z) = x^2 + y^2 + z^2 \). We have seen that in rectangular coordinates we get
\[ \nabla^2 f = 6. \]

We now calculate this in cylindrical and spherical coordinate systems, using the formulas given above.

1. **Cylindrical Coordinates.**
   We have \( x = r \cos \theta \) and \( y = r \sin \theta \) so
   \[ f(r, \theta, z) = r^2 \cos^2 \theta + r^2 \sin^2 \theta + z^2 = r^2 + z^2. \]
   Using the above formula:
   \[ \nabla^2 f = \frac{1}{r} \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} = \frac{1}{r} \left( 2r \right) + \frac{0}{r^2} + \frac{2}{r} = 4 + 2 = 6 \]

2. **Spherical Coordinates.**
   We have \( x = \rho \sin \phi \cos \theta, y = \rho \sin \phi \sin \theta, z = \rho \cos \phi \) and \( \rho = \sqrt{x^2 + y^2 + z^2} \), so
   \[ f(r, \theta, z) = \rho^2. \]
Using the above formula:

\[
\nabla^2 f = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{\rho^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2}
\]

\[
= \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho^2 2\rho \right) + 0 + 0
\]

\[
= \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( 2\rho^3 \right)
\]

\[
= \frac{1}{\rho^2} (6 \rho^2)
\]

\[
= 6.
\]

These three different calculations all produce the same result because \(\nabla^2\) is a derivative with a real physical meaning, and does not depend on the coordinate system being used.
References

1. A brilliant animated example, showing that the maximum slope at a point occurs in the direction of the gradient vector. The animation shows:
   - a surface
   - a unit vector rotating about the point \((1, 1, 0)\), (shown as a rotating black arrow at the base of the figure)
   - a rotating plane parallel to the unit vector, (shown as a grey grid)
   - the traces of the planes in the surface, (shown as a black curve on the surface)
   - the tangent lines to the traces at \((1, 1, f(1, 1))\), (shown as a blue line)
   - the gradient vector (shown in green at the base of the figure)


2. A complete set of notes on Pre-Calculus, Single Variable Calculus, Multi-variable Calculus and Linear Algebra. Here is a link to the chapter on Directional Derivatives.

   Here is a link to the chapter on Curl and Divergence.