2 Topics in 3D Geometry

In two dimensional space, we can graph curves and lines. In three dimensional space, there is so much extra space that we can graph planes and surfaces in addition to lines and curves. Here we will have a very brief introduction to Geometry in three dimensions.

2.1 Planes

Just as it is easy to write the equation of a line in 2D space, it is easy to write the equation of a plane in 3D space.

The point-normal equation of a plane

A vector perpendicular to a plane is said to be *normal to the plane* and is called a *normal vector*, or simply a *normal.*

To write the equation of a plane we need a point $P(x_0, y_0, z_0)$ on the plane and a normal vector $\vec{n} = (a, b, c)$ to the plane.

Let $P = (x_0, y_0, z_0)$ be a point on the plane and $\vec{n}$ be a vector perpendicular to the plane. Then a point $Q(x, y, z)$ lies on the plane,

$\Leftrightarrow$ the vector $\vec{PQ}$ lies on the plane,

$\Leftrightarrow$ $\vec{PQ}$ and $\vec{n}$ are perpendicular,

$\Leftrightarrow$ $\vec{n} \cdot \vec{PQ} = 0$,

$\Leftrightarrow$ $(a, b, c) \cdot (x - x_0, y - y_0, z - z_0) = 0$,

$\Leftrightarrow$ $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$.

DEFINITION. *The point-normal equation of a plane that contains the point $P(x_0, y_0, z_0)$ and has normal vector $\vec{n} = (a, b, c)$ is*

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

EXAMPLE. Let P be a plane determined by the points $A = (1, 2, 3)$, $B = (2, 3, 4)$, and $C = (-2, 0.3)$. Find a vector which is normal to the plane. Find an equation of the plane.

Solution: We need a point on the plane and a normal to the plane. The vector $\vec{AB} \times \vec{AC} = (2, -3, 1)$ is a normal to the plane and we take $A = (1, 2, 3)$ as a point on the plane (you can choose B or C instead of A if you want). The equation on the plane in point-normal form is:

$$2(x - 1) - 3(y - 2) + (z - 3) = 0$$

or equivalently,

$$2x - 3y + z = -1$$

Observe that the coefficients of x, y and z are $(2, -3, 1)$ which is the normal to the plane.

2.2 Lines

Vector equation of a line

To write the vector equation of a line, we need a point $P(x_0, y_0, z_0)$ on the line and a vector $\vec{v} = (a, b, c)$ that is parallel to the line.

Definition. *The vector equation of a line that contains the point $P(x_0, y_0, z_0)$ and is parallel to the vector $\vec{v} = (a, b, c)$ is:*

$$P + t\vec{v} = \vec{r}, \text{ where } t \text{ is scalar.}$$

or,

$$\begin{aligned} (x_0, y_0, z_0) + t(a, b, c) &= (x, y, z) \\ (x_0 + ta, y_0 + tb, z_0 + tc) &= (x, y, z) \end{aligned}$$

Parametric equation of a line

The parametric equation of a line is derived from the vector equation of a line.

Definition. *The parametric equation of a line that contains the point $P(x_0, y_0, z_0)$ and is parallel to the vector $\vec{v} = (a, b, c)$ is:*

$$\begin{aligned} x &= x_0 + ta \\ y &= y_0 + tb \\ z &= z_0 + tc \end{aligned}$$

EXAMPLE. Let L which passes through the points $P(1,1,1)$ and $Q(3,2,1)$. Find a vector which is parallel to the line. Find the vector-equation and parametric equation of the line.

SOLUTION: The vector $\vec{PQ} = (2,1,0)$ is parallel to the line and we take the point $P(1,1,1)$ on the line.
The vector equation of the line:

$$(1,1,1) + t(2,1,0) = (x,y,z)$$

The parametric equations of the line:

$$\begin{aligned} x &= 1+2t \\ y &= 1+t \\ z &= 1 \end{aligned}$$

EXAMPLE. Find the equation of the plane which contains the point $(0,1,2)$ and is perpendicular to the line $(1,1,1) + t(2,1,0) = (x,y,z)$.

2.3 Surfaces

The graph in 3D space of an equation in x, y and z is a surface. Often the graph is too difficut to draw, but here we sketch the graph of a few special types of equations whose graphs are easy to visualize.

Cylindrical surfaces

The graph in 3D space of an equation containing only one or two of the three variables x, y, z is called a *cylindrical surface.*

EXAMPLE. Plot $y = x^2$.

Plot $x^2 + y^2 = 5$.

Quadric Surfaces

The graph in 2D space of a second degree equation in x and y is an ellipse, parabola or hyperbola. In 3D space, the graph of a second degree equation in x, y and z is one of six quadric surfaces.

1. Ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
2. Elliptic Cone $z^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$
3. Elliptic Paraboloid $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$
4. Hyperbolic Parabolid $z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$
5. Elliptic Hyperboloid of one sheet $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
6. Elliptic Hyperboloid of two sheets $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Cross-sections of some quadric surfaces

2.4 Functions of several variables

So far you have studied about functions of one variable, e.g.

$$f(x) = x + x^2.$$

You have learned how to graph these functions, perhaps how to determine their domain and range. You have gone much further in your quest to understand functions; you have learned how to differentiate them, then to calculate the maximum and minimum, then to integrate them. At the end of this term (in March) we will learn how to write a function as a sum of simpler functions, i.e. a Fourier Series Expansion of a function.

However now we will learn something new. We will learn about functions of several variables, e.g.

$$f(x, y) = x^2 + y^2, \quad g(x, y) = 2xy + 7, \quad h(x, y) = e^x + 2y.$$

You will very quickly see that although the concepts are new, the techniques are old and familiar.

Graphs and Level Sets

To draw the graph of $f(x) = x^2$, we drew the graph of the equation

$$y = x^2.$$

Similarly, to draw the graph of the equation $f(x, y) = x^2 + y^2$, we draw the graph of the equation

$$z = x^2 + y^2.$$

We now use the methods developed in the last lecture to draw graphs. Note however that it is difficult to graph general surfaces.

REMARK. The graph of a function $f(x)$ of one variable is the graph of the equation $y = f(x)$, a curve in 2D space. The graph of a function $f(x, y)$ of two variables is the graph of the equation $z = f(x, y)$, a surface in 3D space. The graph of a function $f(x, y, z)$ is a set of points in 4D space and we cannot draw the graph.

One way to understand functions of two or more variables is by using *level sets*.

Example. An example of level sets is a topographic map, which maps hills and valleys in a region by drawing curves indicating height or elevation. If $h(x, y)$ is the height function over a region, say the Himalayas, then if we mark all the points (x, y) on the ground at which the height of the mountain is $h(x, y) = 3000m$, we get the *level set* for $L = 3000$. If we draw the level sets for different heights, e.g 2000m, 3000m, 4000m, 5000m, 6000m, 7000m, 8000m, we get a rough topographic map for the Himalayas.

Note that we draw the level sets on the ground, i.e. in the domain.

Definition. *We fix a number C. The level set of a function of two variables $f(x, y)$ is the set of points (x, y) in the domain which satisfy the equation*

$$f(x, y) = C.$$

For every real number C, we get a level set.

In general, for a fixed number C and a function of several variables f, we define the level set to be the collection of points in the domain which satisfy the equation $f = C$.

Example. Let $f(x) = x^2$ and say the constant $c = 1$. The level set is the set of points such that

$$\begin{aligned} x^2 &= 1 \\ x &= 1, -1 \end{aligned}$$

Hence the level set for $c = 1$ is {1, -1}.

Example. Find the level sets of the function $f(x, y) = x^2+y^2$ for $C = 1, 4, 9$.

Solution: The level sets are

$$\begin{aligned} C = 1 &: x^2 + y^2 = 1; && \text{a circle of radius 1} \\ C = 4 &: x^2 + y^2 = 4; && \text{a circle of radius 2} \\ C = 9 &: x^2 + y^2 = 9; && \text{a circle of radius 3} \end{aligned}$$

EXAMPLE. Let $f(x, y, z) = x^2 + y^2 + z^2$ and say the constant $c = 1$. The level set is the set of points such that

$$x^2 + y^2 + z^2 = 1$$

Hence the level set for $c = 1$ is the set of points lying on the unit sphere.

2.5 Change of Coordinates

Before we describe cylindrical and spherical coordinate systems, we will recall the polar coordinate system in 2D space.

Polar Coordinates

The polar coordinate system is equivalent to the rectangular coordinate system. It locates points using two coordinates r and θ. The coordinate r is the distance from a point to the origin, and θ is the angle used in trignometry which measures the counterclockwise rotation from the positive X-axis.

Conversion from rectangular to polar coordinates:
Let (x, y) be a point in the rectangular coordinate system.

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \tan\theta &= \frac{y}{x} \end{aligned}$$

Conversion from polar to rectangular coordinates:
Let (r, θ) be a point in the polar coordinate system.

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

EXAMPLE. Express in polar coordiantes the portion of the unit disc that lies in the first quadrant.

SOLUTION: The region may be expressed in polar coordinates as

$$0 \le r \le 1; \qquad 0 \le \theta \le \pi/2$$

EXAMPLE. Express in polar coordinates the function

$$f(x, y) = x^2 + y^2 + 2yx.$$

SOLUTION: We substitute $x = r\cos\theta$ and $y = r\sin\theta$ in f, to get

$$f(r, \theta) = r^2\cos^2(\theta) + r^2\sin^2(\theta) + 2r^2\sin(\theta)\cos(\theta) = r^2(1 + \sin(2\theta)).$$

Cylindrical Coordinates

This is a three dimensional extension of plane polar coordinates. Cylindrical coordinates are given by the 3-tuple (r, θ, z), the polar coordinates of the X, Y plane and the rectangular coordinate z. Given a point (x, y, z) in 3-dimensional space, to calculate r and θ we project the point to the XY-plane $(x,\ y,\ z) \mapsto (x,\ y)$. Then calculate $(r,\ \theta)$ as in the previous section.

Cylindrical to Rectangular Conversion Formulas:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}$$

Rectangular to Cylindrical Conversion Formulas:

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \theta &= \tan^{-1}(\frac{y}{x}) \\ z &= z \end{aligned}$$

EXAMPLE. 1. Express in cylindrical coordinates the function

$$f(x, y, z) = x^2 + y^2 + z^2 - 2z\sqrt{x^2 + y^2}$$

2. Express in rectangular coordinates the equation

$$r = \sin\theta$$

Spherical coordinates

Spherical coordinates consist of the 3-tuple (ρ, θ, ϕ). These are determined as follows:

1. $\rho =$ the distance from the origin to the point.
2. $\theta =$ the same angle that we saw in polar/cylindrical coordinates.
3. $\phi =$ the angle between the positive z-axis and the line from the origin to the point.

Spherical to Rectangular Conversion Formulas:

$$\begin{aligned} x &= \rho \sin(\phi) \cos(\theta) \\ y &= \rho \sin(\phi) \sin(\theta) \\ z &= \rho \cos \phi \end{aligned}$$

Rectangular to Spherical Conversion Formulas:

$$\begin{aligned} \rho &= \sqrt{x^2 + y^2 + z^2} \\ \theta &= \tan^{-1}(\frac{y}{x}) \\ \phi &= \cos^{-1}(\frac{z}{\rho}) \end{aligned}$$

Example. 1. Express in spherical coordinates the function

$$f(x, y, z) = x^2 + y^2 + z^2$$

2. Express in rectangular coordinates the equation

$$\rho = 5$$

References

1. A complete set of notes on Pre-Calculus, Single Variable Calculus, Multivariable Calculus and Linear Algebra. Here is a link to the chapter on Lines, Planes and Quadric Surfaces. Also read the section on cylindrical and spherical coordiantes.
   http://tutorial.math.lamar.edu/Classes/CalcIII/3DSpace.aspx.

2. A collection of examples, animations and notes quadric Surfaces.
   Quadric Surfaces.

3. Another gallery of animated and graphical demonstrations of calculus and related topics, from the University of Minnesota.
   http://www.math.umn.edu/%7Erogness/quadrics/.

4. Links to various resources on Calculus.
   http://www.calculus.org/.