Chromatic Turán problems and a new upper bound for the Turán density of $\mathcal{K}_4^-$

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Abstract

We consider a new type of extremal hypergraph problem: given an r-graph $\mathcal{F}$ and an integer $k \geq 2$ determine the maximum number of edges in an $\mathcal{F}$-free, k-colourable r-graph on n vertices.

Our motivation for studying such problems is that it allows us to give a new upper bound for an old Turán problem. We show that a 3-graph in which any four points span at most two edges has density less than $0.32975 < \frac{1}{3} - \frac{1}{280}$, improving previous bounds of $\frac{1}{3}$ due to de Caen [2], and $\frac{1}{3} - 4.5305 \times 10^{-6}$ due to Mubayi [13].

1 Introduction and main results

Given an r-graph $\mathcal{F}$ the Turán number $\mathrm{ex}(n, \mathcal{F})$ is the maximum number of edges in an n-vertex r-graph not containing a copy of $\mathcal{F}$. The Turán density of $\mathcal{F}$ is

$$\pi(\mathcal{F}) = \lim_{n\to\infty} \frac{\mathrm{ex}(n, \mathcal{F})}{\binom{n}{r}}.$$

For 2-graphs the Turán density is determined completely by the chromatic number but for $r \geq 3$ there are very few r-graphs for which $\pi(\mathcal{F})$ is known. (Examples of 3-graphs for which $\pi(\mathcal{F})$ is now known include the Fano plane [3], $\mathcal{F} = \{abc, abd, abe, cde\}$ [12] and $\mathcal{F} = \{abc, abd, cde\}$ [9].)

The two most well-known problems in this area are to determine $\pi(\mathcal{K}_4)$ and $\pi(\mathcal{K}_4^-)$, where $\mathcal{K}_4 = \{abc, abd, acd, bcd\}$ is the complete 3-graph on 4 vertices and $\mathcal{K}_4^- = \{abc, abd, acd\}$ is the complete 3-graph on 4 vertices with an edge removed. For $\pi(\mathcal{K}_4)$ we have the following bounds due to Turán and Chung and Lu [4] respectively

$$\frac{5}{9} \leq \pi(\mathcal{K}_4) \leq \frac{3 + \sqrt{17}}{12} = 0.59359\ldots.$$

Although the problem of determining $\pi(\mathcal{K}_4)$ is an extremely natural question in some respects the problem of determining $\pi(\mathcal{K}_4^-)$ is even more basic since $\mathcal{K}_4^-$ is the smallest 3-graph satisfying $\pi(\mathcal{F}) \neq 0$. Note also that the problem of determining $\pi(\mathcal{K}_4^-)$ can be restated as: determine the maximum density of a 3-graph in which any four vertices span less than three edges. (In this last form the problem is a special case of a question due to Brown, Erdős and Sós [1] asking for the maximum number of edges in an r-graph of order n in which any v vertices span less than e edges. The case $r = e = 3$ and $v = 6$ is the well known $(6, 3)$-problem, see Ruzsa and Szemerédi [15].)

The problem of determining $\pi(\mathcal{K}_4^-)$ has been considered by many people, including Turán [17], Erdős and Sós [7], Frankl and Füredi [10], de Caen [2] and Mubayi [13]. Previously the best bounds known were

$$\frac{2}{7} \leq \pi(\mathcal{K}_4^-) \leq \frac{1}{3} - (4.5305 \times 10^{-6}).$$

The upper bound was proved by Mubayi [13], improving on the upper bound $\pi(\mathcal{K}_4^-) \leq 1/3$ due to de Caen [2]. The lower bound follows from the following construction due to Frankl and Füredi [10].

Let $\mathcal{S}$ be the following 3-graph of order 6 with 10 edges

$$\mathcal{S} = \{124, 234, 346, 456, 126, 256, 135, 145, 235, 136\}.$$

Let $|V| = n$ and suppose that V is partitioned as $V = V_1\dot\cup \cdots \dot\cup V_6$. For such a partition we define $\mathcal{H}_\mathcal{S}$ to be the "blow-up" of $\mathcal{S}$. So $\mathcal{H}_\mathcal{S}$ has vertex set V and edge set

$$\mathcal{H}_\mathcal{S} = \{v_{i_1}v_{i_2}v_{i_3} \mid 1 \leq i_1 < i_2 < i_3 \leq 6,\ i_1i_2i_3 \in \mathcal{S} \text{ and } v_{i_j} \in V_{i_j}\}. \tag{1}$$

If the vertex classes V_i are taken to be as equal as possible in size then this yields a $\mathcal{K}_4^-$-free 3-graph with density greater than $5/18$. Moreover if $\mathcal{R}_\mathcal{S}$ is the 3-graph given by iterating this process, partitioning each V_i and inserting a copy of $\mathcal{H}_\mathcal{S}$ repeatedly, then we obtain a $\mathcal{K}_4^-$-free 3-graph with density approaching $2/7$. (See [10] for details.)

Our main aim in this paper is to prove the following theorem, improving the upper bound for $\pi(\mathcal{K}_4^-)$.

Theorem 1 *The Turán density of $\mathcal{K}_4^-$ satisfies*

$$\frac{2}{7} \leq \pi(\mathcal{K}_4^-) < 0.32975 < \frac{1}{3} - \frac{1}{280}.$$

Our approach involves a new type of extremal problem which we call *chromatic Turán problems*. These are questions of the form: given an r-graph $\mathcal{F}$ and an integer $k \geq 2$ determine the maximum number of edges in an $\mathcal{F}$-free, k-colourable r-graph on n vertices. (Recall that an r-graph is *k-colourable* iff its vertices can be partitioned into k classes none of which contain an edge.) We denote this quantity by $\mathrm{ex}_k(n, \mathcal{F})$.

A simple averaging argument shows that for any r, k, n and $\mathcal{F}$

$$\frac{\mathrm{ex}_k(n+1, \mathcal{F})}{\binom{n+1}{r}} \leq \frac{\mathrm{ex}_k(n, \mathcal{F})}{\binom{n}{r}}$$

and so the corresponding *k-chromatic Turán density* can defined as the limit

$$\pi_k(\mathcal{F}) = \lim_{n\to\infty} \frac{\mathrm{ex}_k(n, \mathcal{F})}{\binom{n}{r}}.$$

One obvious reason why such problems do not seem to have been previously considered is that for 2-graphs they are rather uninteresting.

If G is a 2-graph then the Erdős-Simonovits-Stone theorem determines not only the ordinary Turán density of G but also all of the chromatic Turán densities of G.

Theorem 2 (Erdős–Simonovits–Stone [5],[8]) *If the 2-graph G has chromatic number $\chi(G)$ then*

$$\pi(G) = 1 - \frac{1}{\chi(G) - 1}.$$

Corollary 3 *If G is a 2-graph and $k \geq 2$ then*

$$\pi_k(G) = \begin{cases} 1 - \frac{1}{k}, & k \leq \chi(G) - 1, \\ 1 - \frac{1}{\chi(G)-1}, & k \geq \chi(G). \end{cases}$$

For $r \geq 3$ the problems of determining chromatic and ordinary Turán numbers seem to be genuinely different. An obvious reason for this is that while for a 2-graph H the extremal H-free graphs are not only H-free but also $(\chi(H)-1)$-colourable this does not seem to be the case in general.

The particular chromatic Turán problems which we will consider are those of determining $\pi_2(\mathcal{K}_4^-)$ and $\pi_3(\mathcal{K}_4^-)$. We obtain the following bounds.

Theorem 4 *There exists $\omega_2 > 0$ such that the 2-chromatic Turán density $\pi_2(\mathcal{K}_4^-)$ satisfies*

$$0.25682 < \pi_2(\mathcal{K}_4^-) < \frac{3}{10} - \omega_2.$$

Theorem 5 *There exists $\omega_3 > 0$ such that the 3-chromatic Turán density $\pi_3(\mathcal{K}_4^-)$ satisfies*

$$\frac{5}{18} \leq \pi_3(\mathcal{K}_4^-) < \frac{3 + \sqrt{11/3}}{15} - \omega_3.$$

In the next section we will introduce the key ideas linking ordinary and chromatic Turán densities. In the third section we will prove Theorems 4 and 5. In the final section we prove Theorem 1.

Throughout the remainder of this paper we will write $\pi = \pi(\mathcal{K}_4^-)$, $\pi_2 = \pi_2(\mathcal{K}_4^-)$ and $\pi_3 = \pi_3(\mathcal{K}_4^-)$. For a 3-graph $\mathcal{G}$ with vertex set V and $A \subseteq V$ we let $e(A)$ denote the number of edges of $\mathcal{G}$ contained in A. The degree of a vertex $x \in V$ is denoted by $d_x = \#\{yz \mid xyz \in \mathcal{G}\}$ while the degree of a pair of vertices x, y is denoted by $d_{xy} = \#\{z \mid xyz \in \mathcal{G}\}$.

We will let $\mathcal{F}$ denote a $\mathcal{K}_4^-$-free 3-graph with vertex set V of order n and with $\mathrm{ex}(n, \mathcal{K}_4^-) = m = \eta\binom{n}{3}$ edges. Similarly $\mathcal{F}_k$, $k = 2, 3$, will denote a k-colourable $\mathcal{K}_4^-$-free 3-graph with vertex set V of order n and with $m_k = \mathrm{ex}_k(n, \mathcal{K}_4^-)$ edges.

We take ϵ to denote an arbitrary small positive constant (we will assume $\epsilon < 10^{-10}$). We suppose that n is always sufficiently large that whenever $s \geq n/100$ we have $\mathrm{ex}_k(s, \mathcal{K}_4^-) \leq (\pi_k + \epsilon)\binom{s}{3}$ (for $k = 2, 3$) and $\mathrm{ex}(s, \mathcal{K}_4^-) \leq (\pi + \epsilon)\binom{s}{3}$.

For any value $a > 0$ we will use a' to denote $a + \epsilon$.

2 Ordinary and chromatic Turán densities

Let $\mathcal{F}$ be a $\mathcal{K}_4^-$-free 3-graph with vertex set V of order n and with $\mathrm{ex}(n, \mathcal{K}_4^-) = m = \eta\binom{n}{3}$ edges, as defined above. We count edges in subsets of the vertices of $\mathcal{F}$ of size four. If

$$q_i = \#\{A \in V^{(4)} \mid e(A) = i\}$$

then as $\mathcal{F}$ is $\mathcal{K}_4^-$-free we have

$$m(n-3) = q_1 + 2q_2$$

and

$$q_2 = \sum_{xy \in V^{(2)}} \binom{d_{xy}}{2}.$$

Using the following identity (which holds since every edge contains three pairs of vertices)

$$\sum_{xy \in V^{(2)}} d_{xy} = 3m \tag{2}$$

we obtain

$$mn = q_1 + \sum_{xy \in V^{(2)}} d_{xy}^2. \tag{3}$$

Convexity of $f(x) = x^2$ and (2) then imply that

$$mn \geq q_1 + \frac{9m^2}{\binom{n}{2}}. \tag{4}$$

Now $q_1 \geq 0$ yields

$$m \leq \frac{n^2(n-1)}{18}. \tag{5}$$

Dividing by $\binom{n}{3}$ and taking the limit as $n \to \infty$ gives de Caen's bound $\pi(\mathcal{K}_4^-) \leq 1/3$.

Mubayi's improved upper bound for $\pi(\mathcal{K}_4^-)$ [13] follows from (4) by using supersaturation to give a lower bound for q_1. He used a result of Frankl and Füredi [10] characterizing 3-graphs in which every four points span exactly 0 or 2 edges.

Our improved upper bound for π is achieved by an entirely different approach, although we will also implicitly make use of supersaturation at one point.

Our aim in this section is to prove Lemma 6, giving a lower bound on q_1 in terms of the 3-chromatic Turán density π_3. We will say that $A \in V^{(4)}$ is a *good* 4-set iff A spans exactly one edge. Recall that $\eta = m/\binom{n}{3}$.

Lemma 6 *If $\pi > \pi_3$ then for ϵ sufficiently small and $\pi_3' = \pi_3 + \epsilon$ the number of good 4-sets in $\mathcal{F}$ satisfies*

$$q_1 \geq \frac{2mn(1-\gamma)}{3(2-\mu)} + O(n^3). \tag{6}$$

where $\gamma = \pi_3'/\eta$ and

$$\mu = \gamma - \sqrt{1 - \frac{2\gamma}{3} - \frac{\gamma^2}{3}}.$$

We will assume for the remainder of this section that $\pi > \pi_3$ and that ϵ is sufficiently small that $\gamma = \pi_3'/\eta \leq (\pi_3 + \epsilon)/\pi < 1$.

We start with some simple observations. As before $\mathcal{F}$ is a $\mathcal{K}_4^-$-free 3-graph on n vertices with $m = \eta\binom{n}{3} = \mathrm{ex}(n, \mathcal{K}_4^-)$ edges.

We may assume that for any pair of vertices $x, y \in V$ we have $d_x - d_y \leq n-2$, since if this does not hold then by deleting y and duplicating x we obtain a new $\mathcal{K}_4^-$-free 3-graph on n vertices with at least

$$m + d_x - d_y - (n-2) > \mathrm{ex}(n, \mathcal{K}_4^-)$$

edges. Since

$$\sum_{x \in V} d_x = 3m = 3\eta\binom{n}{3}$$

this implies that if $x \in V$ then

$$d_x = \frac{\eta n^2}{2} + O(n). \tag{7}$$

We count q_1, the number of good 4-sets, by considering pairs of disjoint edges $uvw, xyz \in \mathcal{F}$. For two such edges define

$$q(uvw, xyz) = \#\{\text{good 4-sets amongst } uvwx, uvwy, uvwz, xyzu, xyzv, xyzw\}.$$

For an edge $uvw \in \mathcal{F}$ we then define

$$q(uvw) = \sum_{xyz \in \mathcal{F}} q(uvw, xyz).$$

The following lemma shows how this can be used to count the number of good 4-sets in $\mathcal{F}$.

Lemma 7 *If $\mathcal{F}$ is as above then*

$$\sum_{uvw \in \mathcal{F}} q(uvw) = \eta n^2 q_1 + O(n^5). \tag{8}$$

Proof: The LHS of (8) counts a good 4-set A twice for each unordered pair of edges uvw, xyz such that A is either $uvwx, uvwy, uvwz, xyzu, xyzv$ or $xyzw$. If $A = uvwx$ is a good 4-set with single edge uvw then the number of ways of choosing an unordered pair of edges that count A is simply

$$d_x - \#\{xyz \in \mathcal{F} \mid \{y, z\} \cap \{u, v, w\} \neq \emptyset\} = \frac{\eta n^2}{2} + O(n),$$

using (7). Finally $q_1 = O(n^4)$ implies that (8) holds. □

For the remainder of this section we will attempt to find lower bounds for $q(uvw)$, where $uvw \in \mathcal{F}$, and then use (8) to give a lower bound for q_1.

For $x, y \in V$ we let $E_{xy} = \{z \mid xyz \in \mathcal{F}\}$. The following notation and definitions are all relative to some fixed edge $uvw \in \mathcal{F}$. Let

$$E_{uvw} = E_{uv} \cup E_{uw} \cup E_{vw} \qquad \text{and} \qquad D_{uvw} = V \backslash E_{uvw}.$$

Let $|D_{uvw}| = \delta_{uvw} n$. An edge $xyz \in \mathcal{F}$ is *internal* iff it is contained entirely within either E_{uv}, E_{uw} or E_{vw}. We denote the number of such edges by i_{uvw}.

We call an edge $xyz \in \mathcal{F}$ *bad* iff it is not internal and it does not meet D_{uvw}. An edge which is not bad is said to be *good*. We denote the number of bad edges by b_{uvw}. Let $\mathcal{B}_{uvw}$ be the 3-graph with vertex set E_{uvw} and edge set consisting of all the bad edges of $\mathcal{F}$.

The relationship between estimating q_1 and the 3-chromatic Turán problem enters in our next lemma.

Lemma 8 *If $uvw \in \mathcal{F}$ and $|D_{uvw}| = \delta_{uvw} n$ then $\mathcal{B}_{uvw}$ is 3-colourable with a 3-colouring given by the vertex partition $E_{uv} \dot\cup E_{uw} \dot\cup E_{vw}$. Hence*

$$b_{uvw} \leq ex_3((1 - \delta_{uvw})n, \mathcal{K}_4^-). \tag{9}$$

Moreover any internal edge $xyz \in \mathcal{F}$ is good and satisfies $q(uvw, xyz) \geq 2$.

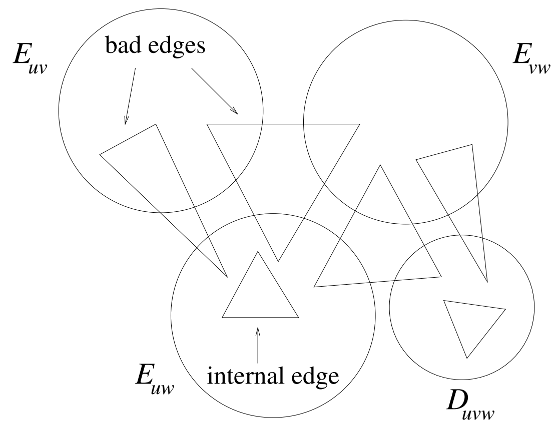

Figure 1: The 3-graph $\mathcal{F}$

Proof: Since $uvw \in \mathcal{F}$ and $\mathcal{F}$ is $\mathcal{K}_4^-$-free so $E_{uv}\dot\cup E_{uw}\dot\cup E_{vw}$ is a partition of E_{uvw}. Moreover since no internal edge belongs to $\mathcal{B}_{uvw}$ this partition yields a 3-colouring of $\mathcal{B}_{uvw}$. Then as $|E_{uvw}| = (1-\delta_{uvw})n$ so (9) holds by definition.

Any internal edge is by definition good so we now need to show that any internal edge xyz satisfies $q(uvw, xyz) \geq 2$.

Let xyz be an internal edge. Without loss of generality we may suppose that $xyz \subseteq E_{uv}$. Now consider the 4-sets $\{xyzu, xyzv, xyzw\}$. Since $\mathcal{F}$ is $\mathcal{K}_4^-$-free and $uvx, uvy, uvz \in \mathcal{F}$ we know that $xyu, xyv, xzu, xzv, yzu, yzv \notin \mathcal{F}$. Hence $xyzu$ and $xyzv$ are both good 4-sets containing the single edge xyz so $q(uvw, xyz) \geq 2$ and the result follows. $\square$

For $W \subseteq V$ let $e_j(W)$ denote the number of edges in $\mathcal{F}$ which contain exactly j vertices from W. We now give a simple lower bound for $q(uvw)$.

Lemma 9 *If $uvw \in \mathcal{F}$ then*

$$q(uvw) \geq 2i_{uvw} + 3e_3(D_{uvw}) + 2e_2(D_{uvw}) + e_1(D_{uvw}) + O(n^2).$$

Proof: We saw in Lemma 8 that if xyz is an internal edge then $q(uvw, xyz) \geq 2$. If $xyz \in \mathcal{F}$ is disjoint from uvw and contains j vertices from D_{uvw} then $q(uvw, xyz) \geq j$, since each vertex in $\{x, y, z\} \cap D_{uvw}$ forms a good 4-set together with uvw. The result then follows since the number of edges meeting uvw is $O(n^2)$. $\square$

We require one final lemma

Lemma 10 *If $uvw \in \mathcal{F}$ and $\pi_3' = \pi_3 + \epsilon$ then*

$$\frac{q(uvw)}{m} \geq 2 - 2\gamma + 3\mu\delta_{uvw} + O\left(n^{-1}\right), \tag{10}$$

where $\gamma = \pi_3'/\eta$ *and*

$$\mu = \gamma - \sqrt{1 - \frac{2\gamma}{3} - \frac{\gamma^2}{3}}.$$

Proof: We will give two lower bounds for $q(uvw)/m$. The first bound (12) is always valid while the second bound (16) is valid only if $\delta_{uvw} \leq 99/100$.

Lemma 9 tells us that

$$\begin{aligned} q(uvw) &\geq 3e_3(D_{uvw}) + 2e_2(D_{uvw}) + e_1(D_{uvw}) + O(n^2) \\ &= \sum_{x \in D_{uvw}} d_x + O(n^2). \end{aligned} \tag{11}$$

Thus for any value of δ_{uvw}, (11), (7) and $|D_{uvw}| = \delta_{uvw} n$ imply that

$$\frac{q(uvw)}{m} \geq 3\delta_{uvw} + O\left(n^{-1}\right). \tag{12}$$

Since

$$m = i_{uvw} + b_{uvw} + e_1(D_{uvw}) + e_2(D_{uvw}) + e_3(D_{uvw})$$

Lemma 9 implies that

$$q(uvw) \geq 2(m - b_{uvw}) + e_3(D_{uvw}) - e_1(D_{uvw}) + O(n^2). \tag{13}$$

Now (9) together with our assumption that $\mathrm{ex}_3(s, \mathcal{K}_4^-) \leq \pi_3' \binom{s}{3}$ for $s \geq n/100$ imply that if $\delta_{uvw} < 99/100$ then

$$b_{uvw} \leq \pi_3' \binom{(1 - \delta_{uvw})n}{3}. \tag{14}$$

Also (7) implies that

$$e_1(D) \leq \sum_{x \in D_{uvw}} d_x = \frac{\eta n^3 \delta_{uvw}}{2} + O(n^2). \tag{15}$$

Let $\gamma = \pi_3'/\eta$. If $\delta_{uvw} \leq 99/100$ then (13), (14) and (15) imply that

$$\frac{q(uvw)}{m} \geq 2 - 2\gamma(1 - \delta_{uvw})^3 - 3\delta_{uvw} + O\left(n^{-1}\right).$$

Expanding we obtain

$$\frac{q(uvw)}{m} \geq 2 - 2\gamma + 3\delta_{uvw}\left(2\gamma - 1 - 2\gamma\delta_{uvw} + \frac{2\gamma\delta_{uvw}^2}{3}\right) + O\left(n^{-1}\right). \tag{16}$$

If

$$\delta_1 = \frac{2(1-\gamma)}{3(1-\mu)}$$

and $\delta_{uvw} \geq \delta_1$ then (12) implies that (10) holds so we may suppose that $\delta_{uvw} \leq \delta_1$. It is easy to check that $\delta_1 \leq 2/3 < 99/100$ and so (16) holds.

To show that (10) holds in this case we need to check that for $\delta_{uvw} \leq \delta_1$ the following inequality holds

$$2\gamma - 1 - 2\gamma\delta_{uvw} + \frac{2\gamma\delta_{uvw}^2}{3} \geq \mu. \tag{17}$$

This is straightforward. Since the LHS of (17) is decreasing in δ_{uvw} it is sufficient to check that

$$2\gamma - 1 - 2\gamma\delta_1 = \mu.$$

Hence (10) holds for all edges $uvw \in \mathcal{F}$. □

Proof of Lemma 6: Let $uvw \in \mathcal{F}$. Since

$$D_{uvw} = \{x \in V \mid x \notin E_{uv} \cup E_{uw} \cup E_{vw}\}$$

and $\delta_{uvw} n = |D_{uvw}|$ we have

$$q_1 = \sum_{uvw \in \mathcal{F}} \delta_{uvw} n. \tag{18}$$

The bound on q_1 in (6) now follows directly from (10) and (8). □

3 Bounds for chromatic Turán problems

Our aim in this section is to give bounds on the chromatic Turán densities of $\mathcal{K}_4^-$. We start by considering the 2-chromatic case. *Proof of Theorem 4:* Let $\mathcal{F}_2$ be a 2-colourable $\mathcal{K}_4^-$-free 3-graph of order n with m_2 edges. Let the two vertex classes of $\mathcal{F}_2$ be A and B, with $|A| = \alpha n$ and $|B| = (1-\alpha)n$. We may suppose that $|A| \leq |B|$ and so $\alpha \leq 1/2$.

Counting edges in 4-sets we obtain an analogous equality to (3)

$$nm_2 = q_1 + \sum_{xy \in A^{(2)} \cup B^{(2)}} d_{xy}^2 + \sum_{xy \in A \times B} d_{xy}^2,$$

where, as previously, q_1 is the number of good 4-sets (that is the number of 4-sets containing exactly one edge). Since neither A nor B contain any edges we have the following two identities

$$\sum_{xy \in A^{(2)} \cup B^{(2)}} d_{xy} = m_2 \quad \text{and} \quad \sum_{xy \in A \times B} d_{xy} = 2m_2.$$

Thus convexity implies that

$$nm_2 \geq q_1 + \frac{m_2^2}{\binom{\alpha n}{2} + \binom{(1-\alpha)n}{2}} + \frac{4m_2^2}{\alpha(1-\alpha)n^2}.$$

Writing $\alpha = (1-\beta)/2$ and using the fact that $q_1 \geq 0$, this implies that $m_2 \leq \frac{n^3}{2f(\beta)}$, where

$$\begin{aligned} f(\beta) &= \frac{4}{(1-\beta)^2 + (1+\beta)^2} + \frac{8}{1-\beta^2} \\ &= \frac{4}{1-\beta^4} + \frac{6}{1-\beta^2} \\ &\geq 10. \end{aligned}$$

Thus we have

$$m_2 \leq \frac{n^3}{20}.$$

Hence dividing by $\binom{n}{3}$ and taking the limit $n \to \infty$ we obtain $\pi_2 \leq 3/10$.

To see that this may be improved to $\pi_2 < 3/10 - \omega_2$ for some $\omega_2 > 0$ we note that we assumed in the above argument that $q_1 = 0$. We can use a supersaturation argument (analogous to that given in [13]) to show that a positive proportion of 4-sets contribute to q_1. (In fact with a little work one can take $\omega_2 > 10^{-4}$ although we will only require $\omega_2 > 0$ in the sequel.) This completes the proof of the upper bound.

For the lower bound we use the following construction. Let $\mathcal{G}_8$ be the 2-colourable $\mathcal{K}_4^-$-free 3-graph of order 8 with the following edges

$$\mathcal{G}_8 = \{125, 135, 145, 126, 136, 246, 346, 456, 127, 237, \\ 247, 357, 457, 367, 138, 238, 348, 258, 268, 178, 478\}$$

Form a blow-up of this 3-graph to give $\mathcal{G}_8(n)$ a 3-graph of order n with vertex classes $V_1, V_2, \ldots, V_8$ of sizes $a_1 n, a_2 n, \ldots, a_8 n$ (so $\sum a_i = 1$) and edges given by

$$\mathcal{G}_8(n) = \{v_{i_1} v_{i_2} v_{i_3} \mid 1 \leq i_1 < i_2 < i_3 \leq 8, i_1 i_2 i_3 \in \mathcal{G}_8 \text{ and } v_{i_j} \in V_{i_j}\}.$$

Now $\mathcal{G}_8(n)$ is clearly still 2-colourable and $\mathcal{K}_4^-$-free. Moreover for the correct choice of $a_1, \ldots, a_8$ and n large it has density greater than 0.25682. (To be precise we can take $a_1 = 0.1608, a_2 = 0.1882, a_3 = 0.1868, a_4 = a_5 = 0.0379, a_6 = 0.1086, a_7 = 0.1437, a_8 = 0.1361$. Such an "optimal" blow-up is found by calculating the Lagrangian of $\mathcal{G}_8$, see for example [11].) □

Turán originally conjectured that $\pi = 1/4$. This was disproved by Frankl and Füredi [10] with their construction of a $\mathcal{K}_4^-$-free 3-graph with $\left(\frac{2}{7} + o(n)\right)\binom{n}{3}$ edges. It is interesting to note that even with the seemingly much stronger condition that $\mathcal{F}_2$ is $\mathcal{K}_4^-$-free and 2-colourable $\mathcal{F}_2$ can still have density greater than 1/4.

We now turn to the the 3-chromatic case and the proof of Theorem 5. This will follow directly from Theorem 4 and the following lemma.

Lemma 11 *The 3-chromatic Turán density of $\mathcal{K}_4^-$ is bounded above by the larger root of*

$$243x^2 - 18x(8\pi_2 + 3) + 64\pi_2^2 = 0. \tag{19}$$

Proof of Theorem 5: The lower bound for π_3 is given by $\mathcal{H}_{\mathcal{S}}$ the 3-graph constructed by Frankl and Füredi which we met earlier (1). Since $\mathcal{H}_{\mathcal{S}}$ is the blow-up of

$$\mathcal{S} = \{124, 234, 346, 456, 126, 256, 135, 145, 235, 136\}$$

a 3-colouring of $\mathcal{S}$ yields a 3-colouring of $\mathcal{H}_{\mathcal{S}}$ in the obvious way. The vertices of $\mathcal{H}_{\mathcal{S}}$ consist of six classes corresponding to the six vertices of $\mathcal{S}$. All the vertices in a single class V_i inherit the colour of the corresponding vertex $i \in V(\mathcal{S})$. A 3-colouring of $\mathcal{S}$ is given by partitioning the vertices as $\{1,2\} \cup \{3,4\} \cup \{5,6\}$. Hence $\mathcal{H}_{\mathcal{S}}$ is 3-colourable and $\mathcal{K}_4^-$-free. It is straightforward to check that it has density at least 5/18.

The upper bound follows by substituting $\pi_2 < 3/10 - \omega_2$ from Theorem 4 into (19) and solving. (It is easy to check that since the bound $\pi_2 \leq 3/10$ yields $\pi_3 \leq (3 + \sqrt{11/3})/15$ so the bound $\pi_2 < 3/10 - \omega_2$ yields $\pi_3 < (3 + \sqrt{11/3})/15 - \omega_3$ for some $\omega_3 > 0$.) □

We note that in this case, unlike the 2-chromatic case, the lower bound could well be the true value.

Using convexity we are able to give a simple lower bound for $\sum d_{xy}^2$ since this is minimized (for $\sum d_{xy}$ constant) by taking all of the degrees to be equal. Our next lemma will allow us to improve this lower bound when some of the pairs $xy \in V^{(2)}$ have smaller than average degree. Lemma 13 then provides a collection of pairs of small degree to which we may apply this result.

Lemma 12 *If $X \subseteq V^{(2)}$, $|X| \geq t$, $\sum_{xy \in V^{(2)}} d_{xy} = S$ and*

$$\frac{1}{|X|} \sum_{xy \in X} d_{xy} \leq \theta \leq \frac{S}{\binom{n}{2}}$$

then

$$\sum_{xy \in V^{(2)}} d_{xy}^2 \geq \theta^2 t + \frac{(S - t\theta)^2}{\binom{n}{2} - t}.$$

Proof: Suppose that $|X| = u \geq t$ and

$$\frac{1}{|X|} \sum_{xy \in X} d_{xy} = \kappa \leq \theta.$$

By the convexity of x^2 we have

$$\begin{aligned} \sum_{xy \in V^{(2)}} d_{xy}^2 &= \sum_{xy \in X} d_{xy}^2 + \sum_{xy \in V^{(2)} \setminus X} d_{xy}^2 \\ &\geq u\kappa^2 + \frac{(S - \kappa u)^2}{\binom{n}{2} - u}. \end{aligned} \tag{20}$$

Now the RHS of (20) is increasing in u and decreasing in κ (for $u \geq t$ and $\kappa \leq \theta$). Hence it is minimized when $\kappa = \theta$ and $u = t$. The result follows. □

Lemma 13 *Let $\pi_2' = \pi_2 + \epsilon$. If $\mathcal{F}_3$ is a $\mathcal{K}_4^-$-free 3-colourable 3-graph of order n with 3-colouring given by the partition $V = A\dot\cup B\dot\cup C$ and $X = A^{(2)} \cup B^{(2)} \cup C^{(2)}$ then*

$$\frac{1}{n|X|}\sum_{xy\in X} d_{xy} \le \frac{8\pi_2'}{9} + O(n^{-1}). \tag{21}$$

Proof: Recall our assumption that n is sufficiently large so that any $\mathcal{K}_4^-$-free 2-colourable 3-graph of order $s \ge n/100$ has at most $\pi_2'\binom{s}{3}$ edges. Let $\mathcal{F}_3$ be as above with 3-colouring given by the partition $V = A\dot\cup B\dot\cup C$, and $|A| \ge |B| \ge |C|$.

We first deal with the case that $|B \cup C|$ is small. So suppose that $|B \cup C| \le n/100$. In this case we have $|X| \ge |A^{(2)}| \ge \binom{99n/100}{2}$. Since $\mathcal{F}_3$ is 3-colourable no edge contains more than one pair from X (otherwise there would be an edge contained in A, B or C) and hence using de Caen's bound (5) we have

$$\begin{aligned}\frac{1}{n|X|}\sum_{xy\in X} d_{xy} &\le \frac{n^2(n-1)}{18n\binom{99n/100}{2}}\\ &< \frac{1}{5}.\end{aligned}$$

So in this case (21) holds since $8\pi_2'/9 > 8\pi_2/9 > 2/9 > 1/5$, by Theorem 4.

We now consider the case that all unions of pairs of vertex classes are reasonably large, so $|B\cup C| \ge n/100$. Let $|A| = \alpha n$, $|B| = \beta n$ so $|C| = (1-\alpha-\beta)n$. We have $99/100 \ge \alpha + \beta \ge 2/3$.

Considering edges containing pairs of vertices from X we obtain

$$\begin{aligned}\frac{1}{n|X|}\sum_{xy\in X} d_{xy} &\le \frac{\pi_2'\left(\binom{(\alpha+\beta)n}{3} + \binom{(1-\alpha)n}{3} + \binom{(1-\beta)n}{3}\right)}{n\left(\binom{\alpha n}{2} + \binom{\beta n}{2} + \binom{(1-\alpha-\beta)n}{2}\right)}\\ &\le \frac{\pi_2'((\alpha+\beta)^3 + (1-\alpha)^3 + (1-\beta)^3)}{3(\alpha^2 + \beta^2 + (1-\alpha-\beta)^2)} + O(n^{-1}).\end{aligned}$$

Thus it is sufficient to prove that

$$\frac{(\alpha+\beta)^3 + (1-\alpha)^3 + (1-\beta)^3}{\alpha^2 + \beta^2 + (1-\alpha-\beta)^2} \le \frac{8}{3}. \tag{22}$$

This is straightforward. Writing $\xi = \alpha + \beta$ and $\rho = \alpha - \beta$ we see that (22) holds iff the following inequality holds

$$0 \le 8 - 28\xi + 30\xi^2 - 9\xi^3 + \rho^2(9\xi - 2).$$

Now $\xi = \alpha + \beta \ge 2/3$ so $9\xi - 2 \ge 4$ and it is sufficient to check that the following inequality holds

$$0 \le 8 - 28\xi + 30\xi^2 - 9\xi^3 + 4\rho^2. \tag{23}$$

The RHS of (23) is clearly increasing in ρ and also in ξ (for $2/3 \le \xi \le 1$). Hence it is minimized at $\rho = 0$ and $\xi = 2/3$ when (23) holds with equality. □

Proof of Lemma 11: Let $\pi_2' = \pi_2 + \epsilon$ and $\mathcal{F}_3$ be a 3-colourable $\mathcal{K}_4^-$-free 3-graph with vertex set V of order $n = 3k$ and of maximum size $m_3 = \mathrm{ex}_3(\mathcal{K}_4^-, n) = \eta_3\binom{n}{3}$. (So $\eta_3 \ge \pi_3$.) Let a 3-colouring of $\mathcal{F}_3$ be given by the partition $V = A\dot{\cup}B\dot{\cup}C$ with $|A| = \alpha n$, $|B| = \beta n$ and $|C| = (1-\alpha-\beta)n$. We may suppose that $|A| \ge |B| \ge |C|$ and hence $2/3 \le \alpha + \beta \le 1$.

We will wish to consider sums over pairs of vertices and so define

$$X = A^{(2)} \cup B^{(2)} \cup C^{(2)}.$$

Note that $|X|$ is minimized when A, B and C are as equal as possible in size. Hence

$$|X| \ge 3\binom{n/3}{2}.$$

Counting edges in 4-sets we obtain an analogous equality to (3)

$$nm_3 = q_1 + \sum_{xy \in X} d_{xy}^2 + \sum_{xy \in V^{(2)} \setminus X} d_{xy}^2, \tag{24}$$

where, as previously, q_1 is the number of good 4-sets (that is the number of 4-sets containing exactly one edge).

Letting $\pi_3' = \pi_3 + \epsilon$ and noting that $\pi_2' \le \pi_3'$, Lemma 13 says precisely that the average degree of pairs of vertices from X is at most

$$\begin{aligned} \frac{8\pi_2' n}{9} + O(1) &\le \frac{8\pi_3' n}{9} + O(1) \\ &< \eta_3(n-2), \end{aligned}$$

for n large (since $\epsilon < 10^{-10}$).

Hence the average degree of pairs of vertices from X is strictly less than the average degree of pairs of vertices from V. (The average degree of pairs of vertices from V being $\eta_3(n-2)$.)

Using $q_1 \ge 0$, $n = 3k$, (24) and Lemma 12 with $\theta = 8\pi_2' n/9 + O(1)$ and $t = 3\binom{k}{2}$ we obtain

$$3km_3 \ge 3\binom{k}{2}\left(\frac{8\pi_2' k}{3}\right)^2 + \frac{(3m_3 - 8\pi_2' k\binom{k}{2})^2}{\binom{3k}{2} - 3\binom{k}{2}} + O(k^3).$$

Dividing by $\frac{k}{18}\binom{3k}{3}$ and rearranging we obtain

$$0 \ge 243\eta_3^2 - 18\eta_3(8\pi_2' + 3) + 64(\pi_2')^2 + O(k^{-1}).$$

Since $\pi_3 \le \eta_3$ and this last inequality holds for all ϵ sufficiently small and $n = 3k$ sufficiently large, the result follows. □

4 A new upper bound for the Turán density of $\mathcal{K}_4^-$

For Theorem 1 we need to show that $\pi < 0.32975$. This will require another new idea, enabling us to not only give a lower bound for q_1 but also to show that if π is close to $1/3$ then the degrees of pairs of vertices in an extremal $\mathcal{K}_4^-$-free 3-graph will not all be equal. To be precise we will show that if π is close to $1/3$ then we can find a collection of pairs of vertices which have lower than average degree and then appeal to Lemma 12 to improve our lower bound for $\sum d_{xy}^2$.

We define $\pi_0 = 1 - 3\pi > 0$. Our next lemma tells us that if π_0 is small (so π is close to $1/3$) then we can find an edge uvw such that the degrees of pairs of vertices from $E_{uv}^{(2)} \cup E_{uw}^{(2)} \cup E_{vw}^{(2)}$ are small.

Lemma 14 *Let $\pi_0' = \pi_0 + \epsilon$ and $\pi_2' = \pi_2 + \epsilon$. There is an edge $uvw \in \mathcal{F}$ such that if $X_{uvw} = E_{uv}^{(2)} \cup E_{uw}^{(2)} \cup E_{vw}^{(2)}$ then*

$$\frac{1}{n|X_{uvw}|} \sum_{xy \in X_{uvw}} d_{xy} \leq \frac{8\pi_2'}{9} + \nu_0 + O(n^{-1}), \tag{25}$$

where

$$\nu_0 = \frac{(2 - 8\pi_2'/9)\pi_0'}{1 - 2\pi_0'}. \tag{26}$$

Furthermore $\delta_{uvw} = |D_{uvw}|/n$ satisfies $\delta_{uvw} \leq \delta_0$ where

$$\delta_0 = \frac{\pi_0'}{(1 - 2\pi_0')} \left(1 + \frac{3}{(1 - 16\pi_2'/9)}\right). \tag{27}$$

Proof: Recall our assumption that n is sufficiently large that any 2-colourable $\mathcal{K}_4^-$-free 3-graph of order $s \geq n/100$ has at most $\pi_2'\binom{s}{3}$ edges, where $\pi_2' = \pi_2 + \epsilon$. Let $uvw \in \mathcal{F}$ and $X_{uvw} = E_{uv}^{(2)} \cup E_{uw}^{(2)} \cup E_{vw}^{(2)}$. As in Section 2 let

$$D_{uvw} = V \backslash (E_{uv} \cup E_{uw} \cup E_{vw}),$$

$$i_{uvw} = \#\{xyz \in \mathcal{F} \mid xyz \subset E_{uv} \text{ or } xyz \subset E_{uw} \text{ or } xyz \subset E_{vw}\}$$

(that is i_{uvw} is the number of internal edges) and $|D_{uvw}| = \delta_{uvw} n$. For $i = 0, 1$ let e_i denote the number of edges in $\mathcal{F}$ meeting D_{uvw} in exactly i vertices and containing exactly one pair from X_{uvw}.

Considering the different types of edges containing a pair of vertices from X_{uvw} we obtain the following identity (see Figure 4)

$$\sum_{xy \in X_{uvw}} d_{xy} = e_0 + e_1 + 3i_{uvw}. \tag{28}$$

We now need to identify a particular choice of edge $uvw \in \mathcal{F}$.

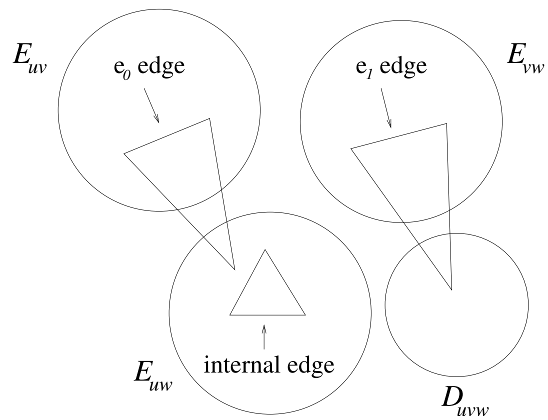

Figure 2: The edges counted by $\sum_{xy\in X_{uvw}} d_{xy}$.

Let $\tau = 1/2 - 8\pi_2'/9$, $\iota_{uvw} = i_{uvw}/m$ and

$$\chi_0 = \min_{xyz\in\mathcal{F}} \left\{ \frac{\iota_{xyz}}{(1-\delta_{xyz})^2} + \tau\delta_{xyz} \right\}.$$

We claim that

$$\chi_0 \le \frac{(2-8\pi_2'/9)\pi_0'}{1-2\pi_0'}, \tag{29}$$

where $\pi_0' = \pi_0 + \epsilon$.

To see this recall Lemma 9 and (7). These imply that for any edge $uvw \in \mathcal{F}$ we have

$$\begin{aligned} q(uvw) &\ge 2i_{uvw} + \sum_{x\in D_{uvw}} d_x \\ &\ge 2i_{uvw} + 3m\delta_{uvw} + O(n^2). \end{aligned}$$

Hence we obtain

$$\frac{q(uvw)}{m} \ge 2\iota_{uvw} + 3\delta_{uvw} + O(n^{-1}).$$

Now for any $uvw \in \mathcal{F}$ the definition of χ_0 implies that

$$2\iota_{uvw} \ge 2(\chi_0 - \tau\delta_{uvw})(1-2\delta_{uvw}).$$

Hence

$$\frac{q(uvw)}{m} \ge 2\chi_0 + \sigma\delta_{uvw} + O(n^{-1}), \tag{30}$$

where $\sigma = 3 - 4\chi_0 - 2\tau$. Lemma 7 tells us that

$$\sum_{uvw\in\mathcal{F}} \frac{q(uvw)}{m} = \frac{6q_1}{n} + O(n^2),$$

while we also have the identity (18)

$$q_1 = \sum_{uvw \in \mathcal{F}} \delta_{uvw} n.$$

Hence (30) implies that

$$q_1 \geq \frac{2\chi_0 mn}{6-\sigma} + O(n^3).$$

Thus (4)

$$mn \geq q_1 + \frac{9m^2}{\binom{n}{2}},$$

implies that for n sufficiently large

$$\frac{2\chi_0}{6-\sigma} \leq \pi_0'.$$

Rearranging this yields (29) proving the claim.

We now choose $uvw \in \mathcal{F}$ such that

$$\frac{\iota_{uvw}}{(1-\delta_{uvw})^2} + \tau\delta_{uvw} = \chi_0. \tag{31}$$

Since $\iota_{uvw} \geq 0$ we have

$$\delta_{uvw}\tau \leq \frac{(2 - 8\pi_2'/9)\pi_0'}{1-2\pi_0'}.$$

Dividing by $\tau = 1/2 - 8\pi_2'/9$ this implies that $\delta_{uvw} \leq \delta_0$, where δ_0 is given by (27). Moreover since $\pi_0' < 1/20$ (as $\pi \geq 2/7$) and $\pi_2' < 3/10$ (by Theorem 4) it is easy to check that $\delta_0 < 1/2$.

We now revisit (28), for which we wish to find an upper bound in the case of $uvw \in \mathcal{F}$ chosen to satisfy (31).

For any vertex $t \in D_{uvw}$ we know that E_t (the neighbourhood of t) is a triangle-free 2-graph. Hence, by Turán's theorem, we have

$$\frac{e_1}{|X_{uvw}|} \leq \frac{\delta_{uvw} n}{2} + O(1).$$

Since e_0 counts the number of edges in a 3-colourable $\mathcal{K}_4^-$-free 3-graph of order $n(1-\delta_{uvw})$ with two vertices in a single vertex class we can bound $e_0/|X_{uvw}|$ using Lemma 13 which implies (since $\delta_{uvw} \leq \delta_0 < 1/2$) that

$$\frac{e_0}{n|X_{uvw}|} \leq \frac{8\pi_2'}{9}(1-\delta_{uvw}) + O(n^{-1}).$$

Since $\tau = 1/2 - 8\pi_2'/9$ and $\iota_{uvw} = i_{uvw}/m$, (28) yields

$$\frac{1}{n|X_{uvw}|} \sum_{xy \in X_{uvw}} d_{xy} \leq \frac{8\pi_2'}{9} + \tau\delta_{uvw} + \frac{3m\iota_{uvw}}{n|X_{uvw}|} + O(n^{-1}).$$

Now

$$\begin{aligned}|X_{uvw}| &= \binom{|E_{uv}|}{2}+\binom{|E_{uw}|}{2}+\binom{|E_{vw}|}{2}\\ &\geq 3\binom{n(1-\delta_{uvw})/3}{2}=\frac{n^2(1-\delta_{uvw})^2}{6}+O(n).\end{aligned}$$

By de Caen's bound $m < n^3/18$. So we have

$$\frac{1}{n|X_{uvw}|}\sum_{xy\in X_{uvw}} d_{xy} \leq \frac{8\pi_2'}{9}+\tau\delta_{uvw}+\frac{\iota_{uvw}}{(1-\delta_{uvw})^2}+O(n^{-1}).$$

Using (29) and (31) this implies that (25) holds. □

Our next lemma tells us that either π_0 is large or there is a non-trivial lower bound for $\sum d_{xy}^2$.

Lemma 15 *Let $\mathcal{F}$ be as before, with $m=\eta\binom{n}{3}$ then either $\pi_0 = 1-3\pi \geq 1/33$ or*

$$\sum_{xy\in V^{(2)}} d_{xy}^2 \geq n^2\binom{n}{2}\left(\eta^2+\frac{\lambda_0^2 x_0}{1-x_0}\right)+O(n^3). \tag{32}$$

Where $\lambda_0=\eta-\nu_0-8\pi_2'/9$, ν_0 is given by (26) and

$$x_0=\frac{(1-\delta_0)^2}{3},$$

with δ_0 given by (27).

Proof: Let $uvw \in \mathcal{F}$ be an edge given by Lemma 14. If $X = X_{uvw} = E_{uvw}^{(2)} \cup E_{uw}^{(2)} \cup E_{vw}^{(2)}$ then $|X| \geq x_0\binom{n}{2}+O(n)$ and

$$\frac{1}{|X|}\sum_{xy\in X} d_{xy} \leq (\eta-\lambda_0)(n-2)+O(1).$$

If $\lambda_0>0$ then we can apply Lemma 12 with $\theta=(\eta-\lambda_0)(n-2)+O(1)$, $t=x_0\binom{n}{2}$ and $S=3m=3\eta\binom{n}{3}$ to yield (32). It remains to show that $\lambda_0>0$.

Since $\pi\leq\eta$ it is easy to check that $\lambda_0>0$ if the following inequality holds

$$\pi(1-2\pi_0')-2\pi_0' > \frac{8\pi_2'}{9}(1-3\pi_0').$$

Now since $\pi_0=1-3\pi$ and $\pi_2'<3/10$ (by Theorem 4) this will hold if

$$10(\pi_0')^2-33\pi_0'+1>0.$$

This last inequality certainly holds if $\pi_0\leq 1/33$ and ϵ is sufficiently small. □

We are now ready to prove Theorem 1.

Proof of Theorem 1: Let $0 < \epsilon < \min\{10^{-10}, \omega_2, \omega_3\}$ where ω_2, ω_3 are given by Theorems 4 and 5 (so $\pi_2' < 3/10$ and $\pi_3' < (3 + \sqrt{11/3})/15$).

We will suppose, for a contradiction, that $\pi \geq 0.32975$. So certainly $\pi > \pi_3' > \pi_3$ holds. If $\pi_0 \geq 0.010751 - \epsilon$ then $\epsilon < 10^{-10}$ implies that $\pi < 0.32975$ so we may suppose that $\pi_0 < 0.010751 - \epsilon$ (and so $\pi_0' < 0.010751$).

From Lemma 6 we have the following lower bound on q_1 (since we are assuming that $\pi > \pi_3$)

$$q_1 \geq \frac{2mn(1-\gamma)}{3(2-\mu)} + O(n^3),$$

where $\gamma = \pi_3'/\eta$ and

$$\mu = \gamma - \sqrt{1 - \frac{2\gamma}{3} - \frac{\gamma^2}{3}}.$$

Now $\eta \geq \pi \geq 0.32975$ and $\pi_3' < (3 + \sqrt{11/3})/15$ imply that $\gamma < \gamma_0 = 0.9936527$. Moreover it is routine to check that the following function is decreasing in γ

$$f(\gamma) = \frac{1-\gamma}{2-\mu}.$$

Hence

$$q_1 \geq \frac{\eta f(\gamma_0) n^4}{9} + O(n^3). \qquad (33)$$

We now consider lower bounds for $\sum d_{xy}^2$. Since $\pi_0 < 1/33$, Lemma 15 implies that (32) holds. The RHS of (32) is increasing in x_0 and λ_0 so we require lower bounds on these quantities.

First consider δ_0, given by (27). This is increasing in π_0' and π_2'. Hence $\pi_0' < 0.010751$ and $\pi_2' < 3/10$ imply that $\delta_0 < 0.08162$. Thus

$$x_0 = \frac{(1-\delta_0)^2}{3} > 0.28114.$$

Writing $\zeta = 8\pi_2'/9$, Lemma 15 and (26) imply that

$$\lambda_0 \geq \pi - \zeta - \frac{(2-\zeta)\pi_0'}{1-2\pi_0'}.$$

The RHS of this last inequality is decreasing in ζ and π_0'. Moreover $\pi_2' < 3/10$ implies that $\zeta < 4/15$. Together with $\pi_0' < 0.010751$ this implies that $\lambda_0 > 0.044038$. Hence (32) implies that

$$\sum_{xy \in V^{(2)}} d_{xy}^2 \geq \frac{n^4}{2}\left(\eta^2 + c_0\right) + O(n^3), \qquad (34)$$

where $c_0 = 0.0007584$. We now use (3) which says that

$$mn = q_1 + \sum_{xy \in V^{(2)}} d_{xy}^2.$$

Combining (3), (33) and (34) we obtain

$$0 \geq \frac{n^4\eta^2}{2} - n^4\eta\left(\frac{1}{6} - \frac{f(\gamma_0)}{9}\right) + \frac{c_0 n^4}{2} + O(n^3).$$

Dividing by n^4 and evaluating we obtain

$$0 \geq 0.5\eta^2 - 0.1660246\eta + 0.0003792 + O(n^{-1}).$$

But now the RHS of this last inequality is increasing in η, thus $\eta \geq \pi \geq 0.329725$ implies that

$$0 \geq 0.0000001 + O(n^{-1}),$$

which clearly cannot hold for n sufficiently large. □

Clearly any improvement in the upper bound for π_2 would directly yield an improvement in the upper bound for π_3, via Lemma 11.

An improvement in the upper bound for π_3 would also yield an improvement in the upper bound for π, although this is more difficult to quantify. Lemma 6 would allow us to obtain an improved lower bound for q_1 which in turn would improve the upper bound for π. However our argument to bound π also involved finding a non-trivial lower bound for $\sum d_{xy}^2$, which relied directly on our upper bound for π_2 (Lemmas 14 and 15).

In the 2-chromatic case we have no real idea as to the true value of π_2 (the construction we have seems very unlikely to yield the correct answer). However in the 3-chromatic case the lower bound is quite possibly correct and we make the following conjecture which would imply a significantly improved upper bound for π.

Conjecture 1 *The 3-chromatic Turán density of $\mathcal{K}_4^-$ is $\frac{5}{18}$.*

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