Invariance under twisting

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Outline

1. The motivation
2. The product
3. The deformation
4. The theorems
### Outline

1. The motivation
2. The product
3. The deformation
4. The theorems
Drinfeld twist

- $H$ bialgebra, $F \in H \otimes H$ a 2-cocycle.
- $H_F$ new bialgebra:
  - Same algebra structure as $H$,
  - Comultiplication $\Delta_F(h) := F\Delta(h)F^{-1}$.
- A an $H$-module algebra.
- $A_{F^{-1}}$ new algebra with $a \ast a' := (G^1 \cdot a)(G^2 \cdot a')$ (being $F^{-1} := G^1 \otimes G^2$).

**Theorem (Majid, 1997)**

$A_{F^{-1}}$ is an $H_F$-module algebra, and

$$A_{F^{-1}} \# H_F \cong A \# H$$
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- \((H, r = r^1 \otimes r^2)\) a f. dim. quasitriangular Hopf algebra.
- \(D(H)\) the Drinfeld double of \(H\):
  - \(D(H) = H^\text{coop} \otimes H\) as a coalgebra.
  - Product \((p \otimes h)(p' \otimes h') := p(h^1 \rightarrow p' \leftarrow S^{-1}(h^3)) \otimes h_2 h'\) (where \(\rightarrow\) and \(\leftarrow\) are the regular actions)
- \(H^*\) a left \(H\)-module algebra structure in \(H^*\) given by
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  h \cdot \varphi := h^1 \rightarrow \varphi \leftarrow S^{-1}(h^2) \\
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"Unbraiding" of braid product

- $(H, r)$ a quasitriangular Hopf algebra
- $H^+, H^- \leq H$ Hopf subalgebras with $r \in H^+ \otimes H^-$
- $B$ a right $H^+$-mod alg. $C$ a right $H^-$-mod alg.
- $B \otimes C$ their braided product wrt $c \otimes b \mapsto br^1 \otimes cr^2$
- $\pi : H^+ \# B \rightarrow B$ alg map with $\pi(1 \# b) = b$

Theorem (Fiore-Steinacker-Wess, 2003)

The map $\theta : C \rightarrow B \otimes C$ given by $\theta(c) := \pi(r^1 \# 1) \otimes cr^2$ is an alg. map from $C$ to $B \otimes C$ and $B \otimes C \cong B \otimes C$. 
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A trivial smash product

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What have these results in common?

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\[ \mathcal{D}(H) \cong H^* \# H \]
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- Two algebras \( X \) and \( Y \)
- A “product” \( Z \) of \( X \) and \( Y \)
- A “deformation” \( \tilde{X} \) of \( X \)
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The Question

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Is it possible to find a general result giving us all the former isomorphisms?

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What do we mean by “product”?

Definition (Cap-Schichl-Vanžura’94, Van Daele’94, . . .)

$Z$ is a **twisted tensor product** of $X$ and $Y$ if there exist a linear map $R : Y \otimes X \rightarrow X \otimes Y$ such that $Z$ is isomorphic to $X \otimes Y$ endowed with the product

$$\mu_R := (\mu_X \otimes \mu_Y) \circ (X \otimes R \otimes Y)$$

Equiv. to conditions given in prof. Schneider’s talk:

- $i_X : X \hookrightarrow Z$ and $i_Y : Y \hookrightarrow Z$ injective algebra maps.
- The map $x \otimes y \mapsto i_X(x) \cdot i_Y(y)$ is a linear isomorphism.

The origin of the story: “Distributive laws”, by J. Beck
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The maps for our examples

All the algebras in our examples are twisted tensor products:

**Drinfeld twist**  \( A \# H = A \otimes_R H \) with \( R(h \otimes a) := h_1 \cdot a \otimes h_2 \).

**Drinfeld double**  \( \mathcal{D}(H) = H^* \otimes_R H \) with
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**Braided product**  \( B \Box C = B \otimes_R C \) with
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**All the rest**  In general, all ordinary tensor products and smash products are twisted tensor products.
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What do we mean by “deformation”?

Informal Definition

By a **deformation** of an algebra $A$ we mean:

- Some datum (maps, other algebras, . . .) associated to $A$
- A new product defined in $A$ upon this datum.

That is, we build a new product, keeping the old vector space.

Remark

This is an **inner deformation**, by contrast to **outer deformations** like Gerstenhaber’s formal deformation.
What do we mean by "deformation"?

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Construction of our deformation I

1. **A, B algebras**
2. \( R : B \otimes A \rightarrow A \otimes B \) linear map
3. Linear maps \( \mu : B \otimes A \rightarrow A \) and \( \rho : A \rightarrow A \otimes B \)
4. Define \( * : A \otimes A \rightarrow A \) by \( * := m_A \circ (A \otimes \mu) \circ (\rho \otimes A) \)
5. Assume the (technical and boring) **compatibility conditions**:
   - \( \rho(1) = 1 \otimes 1 \), \( m_A \circ (A \otimes \mu) \circ (\rho \otimes u_A) = A \)
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   - \( \rho \circ * = (m_A \otimes m_B) \circ (A \otimes R \otimes B) \circ (\rho \otimes \rho) \)

**Theorem**

*The map \( * \) is an associative product in \( A \).*
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The map $*$ is an associative product in $A$. 
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Theorem

The map $\ast$ is an associative product in $A$. 
Construction of our deformation II

Remark
Former datum is a generalization of W. Ferrer and B. Torrecillas \textit{left-right twisting datum}.

Our first two examples fit into this deformation scheme:

Drinfeld twist: \[ \mu(h \otimes a) := h \cdot a, \quad \rho(a) := G^1 \cdot a \otimes G^2. \]
Associated product is \( a * a' = (G^1 \cdot a)(G^2 \cdot a') \), giving \( A_{F-1} \).

Drinfeld double: \[ \mu(h \otimes \varphi) := h_1 \rightarrow \varphi \leftarrow S^{-1}(h_2), \]
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Outline

1. The motivation
2. The product
3. The deformation
4. The theorems
Invariance under twisting: Theorem I

- $A, B$ algebras,
- $(R, \mu, \rho)$ left-right twisting datum with $R$ twisting map.
- $\lambda : A \to A \otimes B$ linear map such that
  - $\lambda(1) = 1 \otimes 1$,
  - $\lambda \circ m_A = (m_A \otimes m_B) \circ (A \otimes \lambda \otimes B) \circ (A \otimes R) \circ (\lambda \otimes A)$
  - $(A \otimes m_B) \circ (\lambda \otimes B) \circ \rho = (A \otimes m_B) \circ (\rho \otimes B) \circ \lambda = A \otimes u_B$
- $A^d$ the deformation of $A$.

**Theorem**

$R^d := (A^d \otimes m_B) \circ (\lambda \otimes m_B) \circ (R \otimes B) \circ (B \otimes \rho)$ is a twisting map, and $(A \otimes m_B) \circ (\rho \otimes B)$ is an algebra isomorphism between $A \otimes_R B$ and $A^d \otimes_{R^d} B$. 
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Consequences

The strong points of our theorem:

- It recovers the isomorphisms in our first two examples.
- The isomorphism is explicitly given.

And the weak ones...

- Last two examples don’t fit.
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- $A \otimes_R B$ a twisted tensor product
- $A'$ another algebra structure on $A$
- $\rho : A' \rightarrow A \otimes_R B$ an algebra map
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The map $R' := (A' \otimes m_B) \circ (\lambda \otimes m_B) \circ (R \otimes B) \circ (B \otimes \rho)$ is a twisting map, and we have an algebra isomorphism

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- This theorem generalizes the former one
- It also contains the last two examples:
  
  **Unbraiding:**  $\lambda(c) := \pi(u^1 \# 1) \otimes c \cdot u^2,$
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  **Trivial smash:**  $\rho(h) = \varphi(1 \# S(h_1)) \otimes h_2,$
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Can this theorem be of any use?

Possible ways of taking advantage of the Invariance Theorem:

- Use it to relate two different twisted tensor products. Could help with the classification, up to isomorphism, of factorization structures.
- Explicitly build a deformation in the terms of the theorem in order to build a new object isomorphic to the original one. Could be used to replace a complicated twisting map by a simpler one.
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Final remarks

1. Most of the results can be translated to (strict) monoidal categories.

2. Under suitable conditions, the Invariance Theorem can be iterated (cf (JLPVO)).

Moral

The study of twisted tensor products allows us to unify apparently unrelated results, proving to be a useful tool in Hopf algebra theory.
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