Handout 12 Gaussian elimination

Notation

We can write a matrix-vector system as an **augmented matrix** with **row sums**:

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ 5 \end{pmatrix} \qquad \text{is equivalent to} \qquad \left(\begin{array}{rrr|r} 1 & 1 & 1 & 6 \\ 2 & 1 & -1 & 1 \\ 1 & -1 & 2 & 5 \end{array}\right) \begin{array}{c} 9 \\ 3 \\ 7 \end{array}$$

Permitted actions

There are only two actions you can do in **standard Gaussian elimination**: they are:

- swap two rows;
- add (or subtract) a multiple of one row to a row below it.

We apply them to **every element in a row** including the "row-sum" number at the end.

Generalised row echelon form

- The first (furthest left) nonzero element of each row is to the right of the first nonzero element of the row above it.
- Any all-zero rows are at the bottom.

Standard Gaussian elimination

We write our system of equations as an augmented matrix (with row sums). Then we choose our first element: a_{11}, so $i = 1$ and $j = 1$ and repeat this process:

1. We have chosen element a_{ij}, i.e. element j of row i. We **ignore all the rows above row** i and **all the columns to the left of column** j from now on.
   - If $a_{ij} = 0$ but there is a non-zero element somewhere below a_{ij} in column j (say, row 4 for instance) then swap rows: $r_i \leftrightarrow r_4$.
   - If the whole of column j from row i down to the bottom is zero then choose $a_{i,j+1}$ as the next element and move on to the next stage.
   - If $a_{ij} \neq 0$ (maybe after the row swap above) then:
     - Subtract a multiple of row i from row $i+1$ to create a zero in place j of row $i+1$;
     - Subtract a multiple of row i from row $i+2$ to create a zero in place j of row $i+2$;
     - Repeat until column j is all zeros below row i.
     - Now choose element $a_{i+1,j+1}$ as the next element and move on to the next stage.
2. Repeat step 1 until we reach generalised row echelon form.

Determinants

Adding rows does not change the determinant of a matrix; swapping a pair of rows multiplies it by (-1). So:

- if our echelon form is an **upper triangular matrix** $\underline{\underline{U}}$ then its determinant is the product of its diagonal elements and our original determinant was $\det(\underline{\underline{A}}) = \pm \det(\underline{\underline{U}})$.
- if our echelon form has a zero on the diagonal, then the original matrix had zero determinant.