A First-order PDEs

First-order partial differential equations can be tackled with the **method of characteristics**, a powerful tool which also reaches beyond first-order. We'll be looking primarily at equations in two variables, but there is an extension to higher dimensions.

A.1 Wave equation with constant speed

Consider the first-order wave equation with constant speed:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0.$$

It responds well to a change of variables:

$$\xi = x + ct \qquad \eta = x - ct$$

The chain rule gives us

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$$

$$\frac{\partial}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta} = c\left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}\right)$$

and so the wave equation is equivalent to

$$2c\frac{\partial u}{\partial \xi} = 0.$$

Integrating gives the general solution $u = F(\eta)$, $u = F(x - ct)$.

But where did we get the change of variables from? The line $x - ct = \text{constant}$ is a line in the x–t plane along which u is constant. This means that if we parametrise this line

$$x = x(r) \qquad t = t(r)$$

then moving along the line by changing r will not change u, i.e.

$$\frac{\mathrm{d}u}{\mathrm{d}r} = 0.$$

This is the underlying principle of the characteristic.

A.2 Variable speed

Let's look now at the variable speed case:

$$\frac{\partial u}{\partial t} + c(x,t)\frac{\partial u}{\partial x} = 0.$$

We would like again to find curves along which u is constant. Suppose such a curve is given by $x = x(r)$ and $t = t(r)$. Then, using the chain rule,

$$\frac{\mathrm{d}u}{\mathrm{d}r} = \frac{\partial u}{\partial t}\frac{\mathrm{d}t}{\mathrm{d}r} + \frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}r}.$$

We want this to be zero, which is easily achieved if we make this expression the same as the original linear operator:

$$\frac{\mathrm{d}t}{\mathrm{d}r}\frac{\partial u}{\partial t} + \frac{\mathrm{d}x}{\mathrm{d}r}\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t} + c(x,t)\frac{\partial u}{\partial x} = 0.$$

This gives us the two parametric equations governing the shape of the characteristic curve:

$$\frac{\mathrm{d}t}{\mathrm{d}r} = 1, \qquad \frac{\mathrm{d}x}{\mathrm{d}r} = c(x,r).$$

These are both ODEs and straightforward to solve.

Example

Look at the equation

$$2\sin\theta\cos 2\phi\frac{\partial u}{\partial\theta} - \frac{\cos\theta\sin 2\phi}{\sin\theta}\frac{\partial u}{\partial\phi} = 0.$$

Suppose that our characteristic is given by $\theta = \theta(r)$, $\phi = \phi(r)$. Then the requirement that u be constant along a characteristic becomes

$$\frac{\partial u}{\partial\theta}\frac{\mathrm{d}\theta}{\mathrm{d}r} + \frac{\partial u}{\partial\phi}\frac{\mathrm{d}\phi}{\mathrm{d}r} = 0.$$

A naïve attempt would be to look at the coupled ODEs

$$\frac{\mathrm{d}\theta}{\mathrm{d}r} = 2\sin\theta\cos 2\phi \qquad \frac{\mathrm{d}\phi}{\mathrm{d}r} = -\frac{\cos\theta\sin 2\phi}{\sin\theta}$$

but we can uncouple them if, before we start, we multiply the original equation by $\sin\theta/\cos\theta\cos 2\phi$:

$$\frac{2\sin^2\theta}{\cos\theta}\frac{\partial u}{\partial\theta} - \frac{\sin 2\phi}{\cos 2\phi}\frac{\partial u}{\partial\phi} = 0,$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}r} = \frac{2\sin^2\theta}{\cos\theta} \qquad \frac{\mathrm{d}\phi}{\mathrm{d}r} = -\frac{\sin 2\phi}{\cos 2\phi}.$$

Now the equations are decoupled, and solving them in turn gives

$$\sin\theta = -\frac{1}{2r} \qquad \sin 2\phi = \exp\left[C - 2r\right].$$

Note that we only use a constant of integration in one of these equations; since r is just a parameter, the point $r = 0$ is not defined *a priori*. Effectively, we are making a change of variables from x, t to r, C. We can invert the transformation:

$$C = \ln\sin 2\phi - \frac{1}{\sin\theta} \qquad r = -\frac{1}{2\sin\theta}$$

and since u is constant on this curve, we can deduce the general solution

$$u = F(C) = F\left(\ln\sin 2\phi - \frac{1}{\sin\theta}\right).$$

A.3 More than two dimensions

Now suppose we have the PDE

$$\frac{\partial u}{\partial x} + c_1(x,y,z)\frac{\partial u}{\partial y} + c_2(x,y,z)\frac{\partial u}{\partial z} = 0.$$

Again, we look for a curve on which u is constant; being a curve, it can still be described with a single variable r so we set $x = x(r)$, $y = y(r)$ and $z = z(r)$. Then the chain rule gives

$$\frac{\mathrm{d}u}{\mathrm{d}r} = \frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}r} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}r} + \frac{\partial u}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}r}$$

and to make this equal to zero we choose

$$\frac{\mathrm{d}x}{\mathrm{d}r} = 1 \qquad \frac{\mathrm{d}y}{\mathrm{d}r} = c_1(x(r), y(r), z(r)) \qquad \frac{\mathrm{d}z}{\mathrm{d}r} = c_2(x(r), y(r), z(r)).$$

The latter two are now coupled ODEs so we are not guaranteed to be able to find a solution; but sometimes you may be lucky.

Example

Look at the equation

$$\frac{\partial u}{\partial x} + xy\frac{\partial u}{\partial y} + 2x^2 z \ln y \frac{\partial u}{\partial z} = 0.$$

We set $x = x(r)$, $y = y(r)$ and $z = z(r)$ and the chain rule gives

$$\frac{\mathrm{d}u}{\mathrm{d}r} = \frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}r} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}r} + \frac{\partial u}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}r}.$$

To match the three coefficients we set:

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}r} &= 1 & x(r) &= r \\
\frac{\mathrm{d}y}{\mathrm{d}r} &= xy = ry & y(r) &= y_0 \exp[r^2/2] \\
\frac{\mathrm{d}z}{\mathrm{d}r} &= 2x^2 z \ln y = -r^4 z \ln y_0 & z(r) &= z_0 \exp[-r^5 \ln y_0/5].
\end{aligned}$$

Now we have expressed all points in terms of the three parameters r, y_0 and z_0 and u is independent of r, so the solution is any function of y_0 and z_0. Reversing the change of variables gives

$$r = x \qquad y_0 = y\exp[-x^2/2] \qquad z_0 = z\exp[-x^7/10]y^{[x^5/5]}$$

and the full solution is

$$u = F(y\exp[-x^2/2]; z\exp[-x^7/10]y^{[x^5/5]}).$$

A.4 Inhomogeneous case

The characteristic curve is just as fundamental if the equation is not homogeneous, although the function value is no longer constant along characteristics. The method is best seen by example:

$$\frac{\partial u}{\partial t} + 2xt\frac{\partial u}{\partial x} = u$$

over all x, with initial condition (at $t = 0$) $u = x$. We start by finding the characteristic. Here the characteristic is given by

$$\frac{\mathrm{d}t}{\mathrm{d}r} = 1 \qquad t = r \qquad \frac{\mathrm{d}x}{\mathrm{d}r} = 2xr \qquad x = x_0 \exp{[r^2]}.$$

This time our two new variables are r and x_0. Along a specific characteristic we have

$$\frac{\mathrm{d}u}{\mathrm{d}r} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} = u \qquad u = u_0 e^r \ \text{ or more exactly } \ u = F(x_0)e^r.$$

We now have a one-parameter family of solutions (parameter x_0): on the curve

$$x = x_0 \exp{[t^2]}, \qquad u = F(x_0)e^t \qquad \text{so} \qquad u = F(x \exp{[-t^2]})e^t.$$

We need to apply the initial conditions to determine the function F. At $t = 0$ we have $x = x_0$ and $u = F(x_0)$ so the initial condition gives $F(x_0) = x_0$:

$$u = x \exp{[-t^2]}e^t = x \exp{[t - t^2]}.$$

A.5 Nonlinear homogeneous case

A general first-order homogeneous PDE in two variables can be written as

$$\frac{\partial u}{\partial t} + c(u, x, t)\frac{\partial u}{\partial x} = 0$$

and the method of characteristics still applies (but we expect an implicit solution in general). The characteristic curves are given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = c(u, x, t).$$

Again, we will write the curve parametrically as $t = r$, and x some function of r and a constant x_0 (that is, constant for a given characteristic). Along any characteristic we will have

$$\frac{\mathrm{d}u}{\mathrm{d}r} = 0$$

and so u is constant along a characteristic. We can then make this into a solution everywhere by setting $u = F(x_0)$ on the characteristic specified by x_0.

Since u is constant on our characteristic, the equation of the curve is simply

$$\frac{\mathrm{d}x}{\mathrm{d}t} = c(F(x_0), x, t),$$

which is a straightforward ODE in x and t. Once we have solved it we have the characteristic curve

$$t = r \qquad x = G(x_0, F(x_0), r) = G(x_0, u, r).$$

The implicit form of the solution is now:

$$x = G(x_0, u, t) \qquad u = F(x_0)$$

which is a one-parameter family with parameter x_0. In many cases it is possible to rearrange the first equation to obtain x_0 in terms of u and t; then substituting this into the second equation gives the more standard form of the implicit solution.

Example

Consider the advection equation

$$\frac{\partial u}{\partial t} + ux^2t\frac{\partial u}{\partial x} = 0.$$

Because it is homogeneous, we expect u to be constant along characteristics: so we parameterise with x_0 (constant on each characteristic) and r (which varies along the characteristic) and we can say $u = F(x_0)$.
Now our characteristic curve becomes

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ux^2t = F(x_0)x^2t,$$

which we can solve:

$$\int \frac{\mathrm{d}x}{x^2} = F(x_0)\int t\,\mathrm{d}t \qquad -\frac{1}{x} = \frac{1}{2}F(x_0)t^2 - \frac{1}{x_0}$$

$$x = \frac{2x_0}{2 - x_0F(x_0)t^2}.$$

Thus the characteristic curve and implicit solution are:

$$t = r \qquad x = \frac{2x_0}{2 - x_0F(x_0)r^2} \qquad u = F(x_0).$$

As above, we can rearrange to get x_0 in terms of x, t and u:

$$x = \frac{2x_0}{2 - x_0ut^2} \qquad x_0 = \frac{2x}{2 + uxt^2}$$

and the standard implicit form of the solution is

$$u = F\left(\frac{2x}{2 + uxt^2}\right).$$

A.6 Nonlinear inhomogeneous

We are now looking at the most complex of first-order PDEs: those of the form

$$\frac{\partial u}{\partial t} + c(u, x, t)\frac{\partial u}{\partial x} = f(u, x, t)$$

Characteristics still exist in these systems, and they may have important physical properties (for instance, discontinuities in the derivatives of the solution will propagate along them) but unfortunately, since u itself now varies along a characteristic, we can no longer solve even implicitly in general.

Example

Here is a case where we can achieve a little:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \cos t.$$

In this case we can immediately spot one solution

$$u = A + \sin t$$

but can we show that this is not the most general solution?
The characteristics are defined by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u;$$

let us suppose we know the family of curves $x = f(x_0, r)$, $t = r$. Then we have

$$\frac{\mathrm{d}u}{\mathrm{d}r} = \frac{\partial u}{\partial t}\frac{\mathrm{d}t}{\mathrm{d}r} + \frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}r} = \frac{\partial u}{\partial t} + \frac{\partial}{\partial r}f(x_0, r)\frac{\partial u}{\partial x}$$

which gives us the two coupled equations

$$\frac{\partial}{\partial r}f(x_0, r) = u \qquad \frac{\mathrm{d}u}{\mathrm{d}r} = \cos r.$$

These are easy to solve in reverse order:

$$u = A(x_0) + \sin r \qquad x = A(x_0)r - \cos r + x_0.$$

Unfortunately, we can't extract x_0 explicitly without choosing the function $A(x_0)$; but note that

$$\begin{array}{lcllll} A(x_0) & = & \alpha & \Longrightarrow & x_0 = x + \cos t - \alpha t & u = \alpha + \sin t \\ A(x_0) & = & x_0/\beta & \Longrightarrow & x_0 = x + \dfrac{\beta \cos t}{t+\beta} & u = \dfrac{x}{\beta} + \dfrac{\cos t}{t+\beta} + \sin t \end{array}$$

so our "spotted" solution is only one of a family of possible solutions.