2 Introduction to perturbation methods

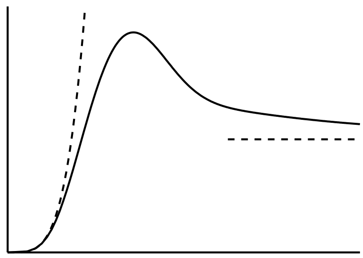
2.1 What are perturbation methods?

Perturbation methods are methods which rely on there being a dimensionless parameter in the problem that is relatively small: $\varepsilon \ll 1$. The most common example you may have seen before is that of high-Reynolds number fluid mechanics, in which a viscous boundary layer is found close to a solid surface. Note that in this case the standard physical parameter Re is large: our small parameter is $\varepsilon = Re^{-1}$.

2.2 A real research example

This comes from my own research[2]. I will not present the equations or the working here: but the problem in question is the stability of a polymer extrusion flow. The parameter varied is wavelength: and for both very long waves (wavenumber $k \ll 1$) and very short waves ($k^{-1} \ll 1$) the system is much simplified. The long-wave case, in particular, gives very good insight into the physics of the problem.

If we look at the plot of growth rate of the instability against wavenumber (inverse wavelength):



we can see good agreement between the perturbation method solutions (the dotted lines) and the numerical calculations (solid curve): this kind of agreement gives confidence in the numerics in the middle region where perturbation methods can't help.

3 Regular perturbation expansions

We're all familiar with the principle of the Taylor expansion: for an analytic function $f(x)$, we can expand close to a point $x = a$ as:

$$f(a+\varepsilon) = f(a) + \varepsilon f'(a) + \tfrac{1}{2}\varepsilon^2 f''(a) + \cdots$$

For general functions $f(x)$ there are many ways this expansion can fail, including lack of convergence of the series, or simply an inability of the series to capture the behaviour of the function; but the paradigm of the expansion in which a

[2] H J Wilson & J M Rallison. *J. Non-Newtonian Fluid Mech.*, **72**, 237–251, (1997)

small change to x makes a small change to $f(x)$ is a powerful one, and the basis of regular perturbation expansions.

The basic principle and practice of the regular perturbation expansion is:

1. Set $\varepsilon = 0$ and solve the resulting system (solution f_0 for definiteness)
2. Perturb the system by allowing ε to be nonzero (but small in some sense).
3. Formulate the solution to the new, perturbed system as a series
$$f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \cdots$$
4. Expand the governing equations as a series in ε, collecting terms with equal powers of ε; solve them in turn as far as the solution is required.

3.1 Example differential equation

Suppose we are trying to solve the following differential equation in $x \geq 0$:

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} + f(x) - \varepsilon f^2(x) = 0, \qquad f(0) = 2. \tag{1}$$

Ignore the fact that we could have solved this equation directly! We'll use it as a model for more complex examples.

We look first at $\varepsilon = 0$:

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} + f(x) = 0, \qquad f(0) = 2, \qquad \Longrightarrow \qquad f(x) = 2e^{-x}.$$

Now we follow our system and set

$$f = 2e^{-x} + \varepsilon f_1(x) + \varepsilon^2 f_2(x) + \varepsilon^3 f_3(x) + \cdots$$

where in order to satisfy the initial condition $f(0) = 2$, we will have $f_1(0) = f_2(0) = f_3(0) = \cdots = 0$. Substituting into (1) gives

$$\begin{array}{ccccccccc}
-2e^{-x} & + & \varepsilon f_1'(x) & + & \varepsilon^2 f_2'(x) & + & \varepsilon^3 f_3'(x) & & \\
+2e^{-x} & + & \varepsilon f_1(x) & + & \varepsilon^2 f_2(x) & + & \varepsilon^3 f_3(x) & & \\
 & - & 4\varepsilon e^{-2x} & - & 4\varepsilon^2 e^{-x} f_1(x) & - & 4\varepsilon^3 e^{-x} f_2(x) & & \\
 & & & & & - & \varepsilon^3 f_1^2(x) & = & O(\varepsilon^4)
\end{array}$$

and we can collect powers of ε:

$$\begin{array}{rcrl}
\varepsilon^0 & : & -2e^{-x} + 2e^{-x} & = 0 \\
\varepsilon^1 & : & f_1'(x) + f_1(x) - 4e^{-2x} & = 0 \\
\varepsilon^2 & : & f_2'(x) + f_2(x) - 4e^{-x} f_1(x) & = 0 \\
\varepsilon^3 & : & f_3'(x) + f_3(x) - f_1^2(x) - 4e^{-x} f_2(x) & = 0
\end{array}$$

The order ε^0 (or 1) equation is satisfied automatically. Now we simply solve at each order, applying the boundary conditions as we go along.

Order ε terms.

$$f_1'(x) + f_1(x) = 4e^{-2x} \qquad \Longrightarrow \qquad f_1(x) = -4e^{-2x} + c_1 e^{-x}$$

and the boundary condition $f_1(0) = 0$ gives $c_1 = 4$:

$$f_1(x) = 4(e^{-x} - e^{-2x}).$$

Order ε^2 terms.

The equation becomes

$$f_2'(x) + f_2(x) = 4e^{-x} f_1(x) \implies f_2'(x) + f_2(x) = 16e^{-x}(e^{-x} - e^{-2x})$$

with solution

$$f_2(x) = 8(-2e^{-2x} + e^{-3x}) + c_2 e^{-x}$$

and the boundary condition $f_2(0) = 0$ gives $c_2 = 8$:

$$f_2(x) = 8(e^{-x} - 2e^{-2x} + e^{-3x}).$$

Order ε^3 terms.

The equation is $f_3'(x) + f_3(x) - f_1^2(x) - 4e^{-x} f_2(x) = 0$ which becomes

$$f_3'(x) + f_3(x) = 48(e^{-2x} - 2e^{-3x} + e^{-4x}).$$

The solution to this equation is

$$f_3(x) = 16(-3e^{-2x} + 3e^{-3x} - e^{-4x}) + c_3 e^{-x}.$$

Applying the boundary condition $f_3(0) = 0$ gives $c_3 = 16$ so

$$f_3(x) = 16(e^{-x} - 3e^{-2x} + 3e^{-3x} - e^{-4x}).$$

The solution we have found is:

$$\begin{aligned} f(x) = 2e^{-x} + 4\varepsilon(e^{-x} - e^{-2x}) &+ 8\varepsilon^2(e^{-x} - 2e^{-2x} + e^{-3x}) \\ &+ 16\varepsilon^3(e^{-x} - 3e^{-2x} + 3e^{-3x} - e^{-4x}) + \cdots \end{aligned}$$

This is an example of a case where carrying out a perturbation expansion gives us an insight into the full solution. Notice that, for the terms we have calculated,

$$f_n(x) = 2^{n+1} e^{-x}(1 - e^{-x})^n,$$

suggesting a *guessed* full solution

$$f(x) = \sum_{n=0}^{\infty} \varepsilon^n 2^{n+1} e^{-x}(1 - e^{-x})^n = 2e^{-x} \sum_{n=0}^{\infty} [2\varepsilon(1 - e^{-x})]^n = \frac{2e^{-x}}{1 - 2\varepsilon(1 - e^{-x})}.$$

Having guessed a solution, of course, verifying it is straightforward: this is indeed the correct solution to the ODE of equation (1).

3.2 Example eigenvalue problem

We will find the first-order perturbations of the eigenvalues of the differential equation

$$y'' + \lambda y + \varepsilon y^2 = 0$$

in $0 < x < \pi$, with boundary conditions $y(0) = y(\pi) = 0$.
[Exercise: repeat this with the final term as εy (easy) or εy^3 (harder).]
First we look at the case $\varepsilon = 0$:

$$y'' + \lambda y = 0$$

This has possible solutions:

$$\begin{array}{ll} \lambda < 0 & y = A\cosh[x\sqrt{-\lambda}] + B\sinh[x\sqrt{-\lambda}] \\ \lambda = 0 & y = Ax + B \\ \lambda > 0 & y = A\cos[x\sqrt{\lambda}] + B\sin[x\sqrt{\lambda}] \end{array}$$

The first two solutions can't satisfy both boundary conditions. The third must have $A = 0$ to satisfy the condition $y(0) = 0$, and the second boundary condition leaves us with

$$B\sin[\pi\sqrt{\lambda}] = 0 \qquad \Longrightarrow \qquad \lambda = m^2, \quad m = 1, 2, \ldots$$

Now we return to the full problem, posing regular expansions in both y and λ:

$$y = \sin mx + \varepsilon y_1 + \cdots$$

$$\lambda = m^2 + \varepsilon\lambda_1 + \cdots$$

Substituting in, we obtain for the differential equation:

$$\begin{array}{rcccccc} -m^2\sin mx & + & m^2\sin mx & & & = & 0 \\ \varepsilon y_1'' & + & \varepsilon m^2 y_1 + \varepsilon\lambda_1\sin mx & + & \varepsilon\sin^2 mx & = & 0 \end{array}$$

As we would expect, the order 1 equation is already satisfied, along with the boundary conditions.

Order ε

The ODE at order ε becomes

$$y_1'' + m^2 y_1 = -\lambda_1\sin mx - \sin^2 mx = -\lambda_1\sin mx + \frac{1}{2}\cos 2mx - \frac{1}{2}.$$

We expect a solution of the form

$$y_1 = A\sin mx + B\cos mx + Cx\cos mx + D\cos 2mx + E$$

and substituting this form back in to the left hand side gives us

$$-2Cm\sin mx - 3m^2 D\cos 2mx + Em^2 = -\lambda_1\sin mx + \frac{1}{2}\cos 2mx - \frac{1}{2}$$

which fixes $C = \lambda_1/2m$, $D = -1/6m^2$, $E = -1/2m^2$. The solution is

$$y_1 = A \sin mx + B \cos mx + \frac{\lambda_1}{2m} x \cos mx - \frac{1}{2m^2} - \frac{1}{6m^2} \cos 2mx.$$

Now we apply the boundary conditions to determine the eigenvalue: $y(0) = 0$ gives

$$0 = B - \frac{1}{2m^2} - \frac{1}{6m^2} \qquad B = \frac{2}{3m^2}$$

and then the condition $y(\pi) = 0$ becomes:

$$0 = \frac{2}{3m^2}(-1)^m + \frac{\lambda_1}{2m}\pi(-1)^m - \frac{1}{2m^2} - \frac{1}{6m^2}$$

which simplifies to determine λ_1:

$$\lambda_1 = \frac{4}{3m\pi}[(-1)^m - 1] = \frac{-8}{3m\pi} \begin{cases} 0 & m \text{ even} \\ 1 & m \text{ odd} \end{cases}$$

Thus the eigenvalues become

$$\lambda = 1 - \frac{8\varepsilon}{3\pi}, 4, 9 - \frac{8\varepsilon}{9\pi}, 16, 25 - \frac{8\varepsilon}{15\pi}, \cdots$$

3.3 Warning signs

As I mentioned earlier, the Taylor series model of function behaviour does not always work. The same is true for model systems: and a regular perturbation expansion will not always capture the behaviour of your system. Here are a few of the possible warning signs that things might be going wrong:

One of the powers of ε produces an insoluble equation
By this I don't mean a differential equation with no analytic solution: that is just bad luck. Rather I mean an equation of the form $x_1 + 1 - x_1 = 0$ which cannot be satisfied by any value of x_1.

The equation at $\varepsilon = 0$ doesn't give the right number of solutions
An nth order ODE should have n solutions. If the equation produced by setting $\varepsilon = 0$ has less solutions then this method will not give all the possible solutions to the full equation. This happens when the coefficient of the highest derivative is zero when $\varepsilon = 0$. Equally, for a PDE, if the solution you find at $\varepsilon = 0$ cannot satisfy all your boundary conditions, then a regular expansion will not be enough.

The coefficients of ε can grow without bound
In the case of an expansion $f(x) = f_0(x) + \varepsilon f_1(x) + \varepsilon^2 f_2(x) + \cdots$, the series may not be valid for some values of x if some or all of the $f_i(x)$ become very large. Say, for example, that $f_2(x) \to \infty$ while $f_1(x)$ remains finite. Then $\varepsilon f_1(x)$ is no longer strictly larger than $\varepsilon^2 f_2(x)$ and who knows what even larger terms we may have neglected...