Birch–Swinnerton-Dyer Study Group: The Parity Conjecture

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Motivation

Let E/K be an elliptic curve over a number field.

Theorem (Mordell–Weil)

$E(K) \cong \mathbb{Z}^{rank(E/K)} \times E(K)_{tors}$

What is $\text{rank}(E/K)$?

Conjecture (BSD 1)

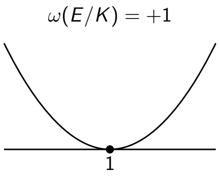
- *$L(E/K, s)$ has analytic continuation to $\mathbb{C}$*
- *$rank(E/K) = ord_{s=1} L(E/K, s)$*

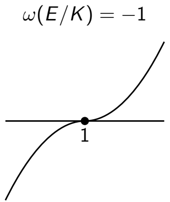
Conjecture (Functional equation)

$L(E/K, s) = \pm L(E/K, 2-s) \cdot (\textit{stuff})$

The sign is called the global root number and is the product of local root numbers:

$$\omega(E/K) = \prod_{\nu} \omega_{\nu}(E/K)$$





The Parity Conjecture

Conjecture (Functional equation)

$L(E/K, s) = \pm L(E/K, 2-s) \cdot (\mathit{stuff})$

The sign is called the global root number and is the product of local root numbers:

$$\omega(E/K) = \prod_{\nu} \omega_\nu(E/K)$$

Note that $\omega(E/K) = (-1)^{\mathrm{ord}_{s=1} L(E/K,s)}$, leading to:

Conjecture (The Parity Conjecture)

$(-1)^{rank(E/K)} = \omega(E/K)$

The p-Parity Conjecture

Write

$$\text{Ш}(E/K)[p^\infty] = \{\text{elements of order } p^n, n \geq 0\}$$
$$= (\mathbb{Q}_p/\mathbb{Z}_p)^{r_p} \times (\text{finite } p\text{-group of } \square \text{ order})$$

Define

$$\text{rank}_p(E/K) = \text{rank}(E/K) + r_p$$
$$= \dim_{\mathbb{Q}_p}(\text{Hom}_{\mathbb{Z}_p}(\varinjlim \text{Sel}_{p^n}(E/K), \mathbb{Q}_p/\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$$

Conjecture (The p-Parity Conjecture)

$(-1)^{rank_p(E/K)} = \omega(E/K)$

Current Status

Conjecture (The Parity Conjecture)

$(-1)^{rank(E/K)} = \omega(E/K)$

- (Assuming $|Ш| < \infty$) Proved for E/K, where K is a number field.

Conjecture (The p-Parity Conjecture)

$(-1)^{rank_p(E/K)} = \omega(E/K)$

- Proved for $E/\mathbb{Q}$.
- Proved (in most cases) for E/K, where K is a totally real field.
- Proved for E/K with a p-isogeny.

The Local Root Number

The local root number is defined using the theory of "local" ϵ-factors.

Theorem (Langlands–Deligne)

There is a unique definition of local ϵ-factors satisfying the following:

- *Multiplicativity*
- *Inductivity in degree 0*
- *Quasi-characters*
- *Semi-simplification*

Letting $K = \mathbb{Q}$:

Definition

$\omega_p(E/\mathbb{Q}) = \frac{\epsilon(\rho)}{|\epsilon(\rho)|}$ where $\rho = (V_l(E))^* \otimes_{\mathbb{Q}_l} \mathbb{C}$

Example 1

Theorem

$$\omega_\nu(E/K) = \begin{cases} -1 & K_\nu = \mathbb{R} \text{ or } \mathbb{C} \\ +1 & E/K_\nu \text{ has good reduction} \\ -1 & E/K_\nu \text{ has split multiplicative reduction} \\ +1 & E/K_\nu \text{ has non-split multiplicative reduction} \end{cases}$$

Let $K = \mathbb{Q}$ and $E : y^2 + xy = x^3 + x^2 - 95x - 399$, (114.a1)

Have $\Delta(E/\mathbb{Q}) = 2^2 \cdot 3^5 \cdot 19$ and $\text{rank}(E/\mathbb{Q}) = 0$

- There is one infinite place
- When $p = 2$ or 3, reduction type is non-split multiplicative
- When $p = 19$, reduction type is split multiplicative

$$\omega(E/K) = -1 \cdot +1 \cdot +1 \cdot -1 = +1$$

Example 2

Now let $K = \mathbb{Q}(\sqrt{19})$ and $E : y^2 + xy = x^3 + x^2 - 95x - 399$, (114.a1)

- There are 2 infinite places
- There is 1 prime above 2, reduction is non-split multiplicative
- There are 2 primes above 3, reduction is non-split multiplicative
- There is 1 prime above 19, reduction is split multiplicative

$$\omega(E/K) = (-1)^2 \cdot +1 \cdot (+1)^2 \cdot -1 = -1$$

Have

$$\mathrm{rank}(E/K) = \mathrm{rank}(E/\mathbb{Q}) + \mathrm{rank}(E_{19}/\mathbb{Q})$$

where

$$E_{19} : y^2 = x^3 - x^2 - 551728x - 157527872 \text{ and } \mathrm{rank}(E_{19}/\mathbb{Q}) = 1$$